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26 Lecture 26

Definition 26.1. *A facet of a polytope P is a codimesnsion of one face of P.*

Lemma 26.2. *Suppose $P \subseteq \mathbb{R}^n$ is a convex polytope and $M = P/\sim$ where $x{\sim}y$ is an equivalence relation on ∂P generated by gluing facets by isometries. Then M is Hausdorff and second countable.*

Proof. Exercise. □

Remark 26.3. M is not necessarily a manifold. For example, consider the cube in Figure 1.

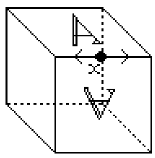

Figure 1

In $M = P/{\sim}$ the point x has no neighbourhood homeomorphic to $\mathbb{R}^3$. Hence is M not a manifold.

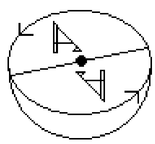

Figure 2

In fact every sufficiently small neighbourhood of x is homeomorphic to a cone on $\mathbb{RP}^2$ and not $\mathbb{S}^2$.

Question 26.4. How can we give the I-bundle map $p : \mathbb{M}^2 \to \mathbb{S}^1$ in coordinates?

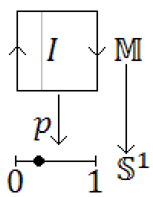

Figure 3

Define $\mathbb{S}^1 \cong I/0{\sim}1$ and $\mathbb{M}^2 \cong \mathbb{I}^2/(1,y){\sim}(0,1-y)$. We set $p(x,y) = x$.

Exercise 26.5. Show this is continuous and gives the desired I-bundle over $\mathbb{S}^1$.

26.1 Fundamental groups of $\mathbb{S}^1$-bundles over surfaces with boundary

Recall if $p : T \to F$ is an $\mathbb{S}^1$-bundle over F a compact connected surface with $\partial F \neq \phi$ then F is a disc with bands as in figure 4.

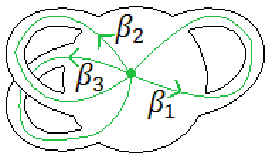

Figure 4

The loops $\{\beta_i\}$ generate $\pi_1(F, x)$ as a free group.
Thus $\pi_1(F, x) \cong \mathbb{F}_k \cong \langle \beta_1, ..., \beta_k \rangle$.

Exercise 26.6. Linked twisted bands can be slid apart.

Proposition 26.7. *Let $x' \in T$, γ and $S(\beta_1)$ generate $\pi_1(T, x')$. In fact, $\pi_1(T, x') \cong \mathbb{Z} \rtimes \pi_1(T, x) \cong \langle \gamma \rangle \rtimes \langle \beta_1, ..., \beta_k \rangle$ where we have the relation $\beta \gamma \beta^{-1} = \gamma^{\tau(\beta)}$.*

$$\begin{array}{ccc} \pi_1(F, X) & \longrightarrow & Aut(\mathbb{Z}) \cong \mathbb{Z}_2 \\ \Cup & & \Cup \\ \beta & \longmapsto & \tau_p(\beta) = \pm 1 \end{array}$$

Figure 5

Question 26.8. How do we know a section exists?
Answer: Because $\partial F \neq \phi$

Exercise 26.9. The Hopf bundle $\mathbb{S}^3 \to \mathbb{S}^2$ has no sections.

Notation 26.10. Suppose $d_1, ..., d_n$ are the boundary components of F. Define $\delta_i = S(d_i)$ this is a closed surface in ∂T. Observe that ∂T is a union of tori and Klein bottles.

Remark 26.11. Consider $T = \mathbb{S}^1 \times (\mathbb{T}^2 - n(x))$ where $n(x)$ is a neighbourhood of point $x \in \mathbb{T}^2$. See Figure 6.

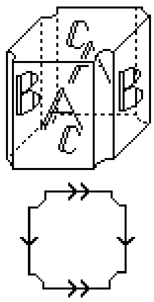

Figure 6

We find that $\partial T \cong \mathbb{T}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$. If T is oriented then ∂T contains no Klein bottles.

26.2 Tori and slopes

Definition 26.12. *Suppose $\mathbb{T}^2$ is a torus. Any simple closed curve $\mu \subset \mathbb{T}^2$ that is non-trivial in $\mathbb{T}^2$ is called a slope.*

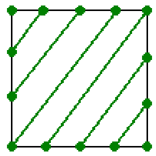

Figure 7

In Figure 7, μ is the slope $4/3$. In Figure 8 we show a few special slopes.

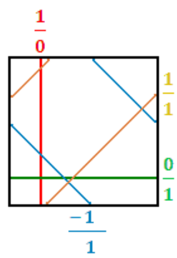

Figure 8

Observe that there is a bijection between {slopes up to isotopy} and $\mathbb{Q} \cup \{1/0\}$.

Exercise 26.13. Check that the set of slopes is in bijection with $\mathbb{Q} \cup \{1/0\}$.

Exercise 26.14. If μ, ν have slopes p/q and r/s respectively then the minimal intersection of μ and ν is $|\mu \cap \nu| = |ps - qr|$.

Example 26.15. (Fibred solid tori) Suppose $U = T(p,q)$ so $U \cong \mathbb{S}^1 \times \mathbb{D}^2$. However, the homeomorphism does not preserve fibers. $M \cong D^2$ comes to us equipped with three interesting slopes.
1) The meridional slope: $\mu = \{pt\} \times \partial\mathbb{D}^2$ unique slope that bounds a disc.
2) The fiber slope: denoted ϕ.
3) The product slope: $\pi = \mathbb{S}^1 \times \{$ point of $\partial\mathbb{D}^2\}$.
The slopes μ, ϕ are canonical.However π involves making a choice of homeomorphism $U \cong \mathbb{S}^1 \times \mathbb{D}^2$.