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27 Lecture 27

27.1 Slopes on Tori

Suppose $U = T(p, q)$, is the fibred solid torus with *orbit invariant* (p, q). Choose a homeomorphism $U \cong S^1 \times \mathbb{D}^2$, this is not canonical. However two such choices only differ (after isotopy) by a *Dehn twist* about the meridional disc.

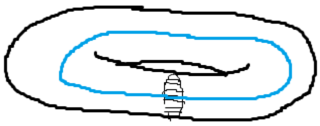

Figure 1

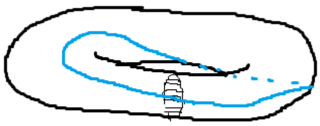

Figure 2

That is, choosing the homeomorphism between U and $S^1 \times \mathbb{D}^2$ amounts to choosing the number of times the blue curve twists around the meridional slope.

Definition 27.1. Let $\mu = pt \times \partial\mathbb{D}^2$ be a *meridional slope.*

Definition 27.2. Let $\lambda = S^1 \times x$ for x in $\partial\mathbb{D}^2$ be the *longitudinal slope.*

Definition 27.3. Let $\phi \subseteq \partial U$ be a fibre, we call ϕ the *fibre slope.*

Figure 3: Here λ twists round three times.

27.2 Algebraic Intersection

Definition 27.4. Suppose $\alpha, \beta \subset \mathbb{T}^2$ are oriented slopes. Define $\alpha \cdot \beta$, the *algebraic intersection* of α with β (in that order) by $\alpha \cdot \beta = \sum_{x \in \alpha \cap \beta} \text{Sign}(x)$, where $\text{Sign}(x) = \pm 1$ as the crossing of α, β at x agrees/disagrees with the orientation of $\mathbb{T}^2$.

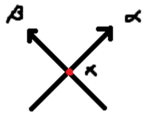

Figure 4: $\mathrm{Sign}(x) = +1$

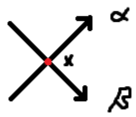

Figure 5: $\mathrm{Sign}(x) = -1$

According to our conventions, $\mu \cdot \lambda = +1$, note that $\lambda \cdot \mu = -1$ and this algebraic intersection is antisymmetric for all α, β.

Example 27.5. Consider $T(1,3)$: $\lambda \cdot \phi = +p$ and $\phi \cdot \lambda = +q$,

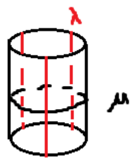

Figure 6

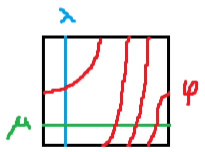

Figure 7

Remark 27.6. Really $\phi \cdot \lambda \equiv q \pmod{p}$ but we can Dehn twist to get exactly q.

Exercise 27.7. The slopes α and β with actual slope represented by $\frac{p}{q}$ and $\frac{r}{s}$ have $|\alpha \cdot \beta| = |ps - qr|$.

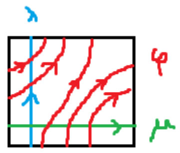

Figure 8

27.3 Seifert Fibred Spaces

Suppose $p\colon M \to B$ is a Seifert bundle, so $B = M/S^1$. Assume that M is orientable and all fibres of M can be consistently oriented. Let $x_i \subseteq$ be a finite set that is nonempty and that contains all of the cone points. Let $D_i = N(x_i)$ be a small disc about x. Let $F = \overline{B - \sqcup D_i}$, this is a surface with $\partial F \neq \emptyset$.

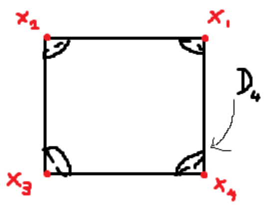

Figure 9

Let $T = M|_F$, so $p\colon T \to F$ is an S^1–bundle over a surface with nonempty boundary. Let $U_i = M|_{D_i}$, so $M = T \cup (\sqcup U_i)$, M is a union of an S^1–bundle with many solid tori.

Let $s\colon F \to T$ be any section for $p\colon T \to F$. Note that this is making a choice. Let $d_i = \partial D_i$ and $\delta_i = s(d_i)$. Notice that δ_i is a slope on ∂U_i, call this the *sectional slope.*

Definition 27.8. $a_i = \mu_i \cdot \phi_i = p_i$, $b_i = \delta_i \cdot \mu_i$. Scott calls $(a_i b_i)$ the *Seifert invariants* of the fibre $C_i = M|_{x_i} = p^{-1}(x_i)$, however these aren't really invariants since b_i depends on the choice of section.

Exercise 27.9. For all i, $\delta_i \cdot \phi_i = \pm 1$, $+1$ in ∂T and -1 in ∂U_i. δ_i and ϕ_i intersect once and so they generate $\mathbb{Z} \oplus \mathbb{Z}$.

Lemma 27.10. $b_i \cdot q_i \equiv 1 (mod p_i)$

Proof. $1 = \mu \cdot \lambda = (a\delta + b\phi) \cdot \lambda = a\delta \cdot \lambda + b\phi \cdot \lambda = p(\delta \cdot \lambda) + b \cdot q$.
So $1 = b \cdot q \pmod p$ as desired. □

Exercise 27.11. $\alpha \cdot \beta = -\beta \cdot \alpha$ thus $\alpha \cdot \alpha = 0$.

Check each, both should follow directly from the definition and so the above justifies writing $\mu = a\delta + b\phi$.