28 Lecture 28

28.1 From last lecture

Recall If M is a Seifert fibered space with base space B, let $x_i \subseteq B = \frac{M}{S^1}$ be a finite collection of points which contains all the cone points of B. Assume M is orientable, so B has no mirror boundary.

Let $D_i = N(x_i)$, $U_i = \frac{M}{D_i}$, $F = \overline{B - \amalg D_i}$

Let μ_i, λ_i, φ_i, δ_i be the meridional, longitudinal, fiber and sectional slopes respectively in $\delta(U_i) \cong \mathbb{T}^2$.

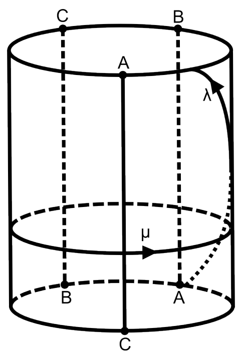

Figure 1: The solid fibred torus U_i. φ is the slope passing through A, B and C.

Let $\alpha \circ \beta$ denote the algebraic intersection of slopes α and β. Notice that $U_i \cong T(p_i, q_i)$ so

1. $\mu_i \circ \lambda_i = 1$ This defines λ_i up to Dehn twists of U_i, i.e λ_i is defined so it meets μ_i exactly once. We make a choice of the number of Dehn twists using $\phi_i \circ \lambda_i = q_i + kp_i$ and this defines λ_i uniquely. (Using k = 0 makes this nice.)
2. $\delta_i \circ \lambda_i = 1$ We have chosen the orientation of the sectional slope δ_i to make this equation true. (δ_i looks like μ_i).
3. $\mu_i \circ \phi_i = p_i$

Definition 28.1. Define a_i and b_i by $a_i = \mu_i \circ \lambda_i$, $b_i = \delta_i \circ \mu_i$.

Hence $a_i = p_i$.

Note $\delta(U) \cong \mathbb{T}^2$ has basis (δ, ϕ). With respect to this basis, we write $\mu = a_i \circ \delta_i + b_i \circ \phi_i$. Let's check this.

$$\begin{aligned}\delta_i \circ \mu_i &= \delta_i \circ (a_i\delta_i + b_i\mu_i) \\ &= a_i(\delta_i \circ \delta_i) + b_i(\delta_i \circ \phi_i) \\ &= b_i\end{aligned}$$

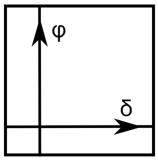

Figure 2: Basis of $\delta(U_i)$

Also

$$\begin{aligned}\mu_i \circ \phi_i &= (a_i\delta_i + b_i\phi_i) \circ \phi_i \\ &= a_i(\delta_i \circ \phi_i) + b_i(\phi_i \circ \phi_i) \\ &= a_i\end{aligned}$$

Lemma 28.2. $b_i \circ q_i \equiv 1 (mod\, p_i)$

Proof done last lecture.

28.2 Euler number

The Euler number (p_i, q_i) is the orbit invariant of U_i. The Seifert invariant is (a_i, b_i). (This is a bad name, as (a_i, b_i) is not actually an invariant.)

These depend on the choice of section $p\colon T \to F$.

Definition 28.3. $e(M) = -\sum \frac{b_i}{a_i}$

Theorem 28.4. *e(M) is an invariant of $p\colon T \to F$, i.e e(M) does not depend on the choice of section.*

Recall There is a sentence in my notes here that does not make any sense. It goes like this: The orientability of B, M and of the fibers, as well as the isomorphism class of B, as well as the isomorphism invariants of $p\colon T \to F$.

Example 28.5. $e(S^1 \times F) = 0$

Exercise 28.6. Prove this directly from the definition.

Example 28.7. $e(Hopf\colon S^3 \to S^1) = 1$

Theorem 28.8. $e(M) = 0$ *if and only if M is finitely covered by* $\widetilde{M} = S^1 \times F$.

Theorem 28.9. *Suppose there is a covering of Seifert fibered structures $M \to N$, of degree d_M. Consider the diagram:*

$$\begin{array}{ccccc} S^1 & \to & M & \to & B \\ \downarrow d_F & & d_M \downarrow & & d_B \downarrow \\ S^1 & \to & N & \to & C \end{array}$$

Then we have that the map $B \to C$ is an orbifold covering of degree d_B and

1. $d_M = d_B \cdot d_F$

2. $d_F \cdot e(M) = e(N) \cdot d_B$

Compare (2) to our previous formula $\chi^{orb}(B) = d_B \cdot \chi^{orb}(C)$. Theorem 28.8 is a weaker version of (2).

In summary: $\chi^{orb}(B)$ and $e(M)$ together suffice to detemine the kind of geometry that models M.

Following Scott:

	$\chi^{orb}(B) < 0$	$\chi^{orb}(B) = 0$	$\chi^{orb}(B) < 0$
$e(M) = 0$	$\mathbb{R} \times S^2$	$\mathbb{E}^3$	$\mathbb{R} \times \mathbb{H}^2$
$e(M) \neq 0$	S^3	Nil	$\mathrm{PSL}(2, \mathbb{R})$

So if $e(M) = 0$ we get products, i.e $S^1 \times F$ where F is a surface, and if $e(M) \neq 0$ we get twisted geometries.