5 Lecture 5

5.1 Group Actions

Definition 5.1. Suppose X is a topological space and G is a group acting on X by homeomorphisms.
(i) Say that G acts *faithfully* if the natural map $G \to Homeo(X)$ is injective.

(ii) For $g \in G$, define $fix(g) := \{x \in X \mid g(x) = x\}$.

(iii) $g \in G$ acts *freely* if $fix(g) = \emptyset$ and the group G is said to act freely if for all $g \in G - \{Id_X\}$, g acts freely.

(iv) G acts *properly discontinuously* if for all $C \subseteq X$ compact, the set $\{g \in G \mid g(C) \cap C \neq \emptyset\}$ is finite.

(v) If $x \in X$ let $Stab(x, G) = \{g \in G \mid g(x) = x\}$. (*Note:* This is a subgroup of G, sometimes called the *isotropy group* at X).

(vi) Suppose X is a geometry and $G < Isom(X)$ is a subgroup (using the notation $<$ to denote subgroup). G is *discrete* (or *acts discretely*) if G is discrete with the subspace topology.

Exercise 5.2. : If X a geometry then $G < Isom(X)$ is discrete iff G acts properly discontinuously.

The topology on $Isom(X)$ is the *compact open topology* (the definition of which is left as a reading exercise).

Example 5.3. $\mathbb{E}^1 = (\mathbb{R}, dx)$.

Figure 5.1: $\mathbb{E}^1$

Exercise 5.4. : $Isom(\mathbb{E}^1) \cong \mathbb{R} \rtimes O(1) = \mathbb{R} \rtimes \mathbb{Z}/2\mathbb{Z}$. (Recall that $O(n) = \{A \in M(n,n) | \ AA^T = I_n\}$).

$Isom(\mathbb{E}^1) \cong \{x \to \pm x + a | \ a \in \mathbb{R}\}$ with translations $x \to x + a$ (denoted by T_a) and reflections $x \to 2a - x$ (denoted by R_a).

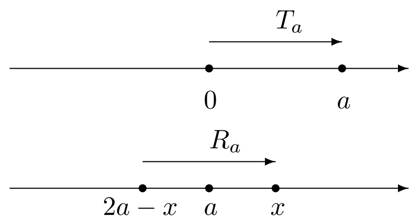


Figure 5.2: Action of T_a and R_a on $\mathbb{E}^1$

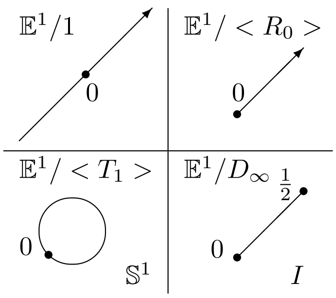
Note that if $a \neq 0$, T_a acts freely and $fix(R_a) = \{a\}$.

Remark 5.5. The subscripts of T_a and R_a are giving parametrisations of the two components of $Isom(\mathbb{E}^1)$.

Exercise 5.6. : The following diagram classifies the discrete subgroups of $Isom(\mathbb{E}^1)$, up to isomorphism.

	freely	not freely
finite	$1 =< T_0 >$	$\mathbb{Z}/2\mathbb{Z} \cong< R_0 >$
not finite	$\mathbb{Z} \cong< T_1 >$	$D_\infty \cong< R_0, R_{\frac{1}{2}} >$

For each G listed above, we have a quotient $\mathbb{E}^1/G$:



Notation: $< A >$ is the group generated by elements of A.

Claim: $D_\infty \cong < \alpha, \beta \mid \alpha^2, \beta^2 > = < A \mid R >$ where A is the set of generators and R are relations.

Note that if w is any word in α, β, for example $w = \alpha^3\beta^3\alpha^{-4}\beta^2\alpha^2\beta^{-3}$, then we may rewrite it using the relations to get $w = \alpha\beta\beta = \alpha$. So any word in $< \alpha, \beta \mid \alpha^2, \beta^2 >$ may be rewritten so that all powers are one and α is followed only β and conversely:
α, β, $\alpha\beta$, $\beta\alpha$, $\alpha\beta\alpha$, $\beta\alpha\beta$... ie the list

$$\left\{ \begin{array}{ll} (\alpha\beta)^n\alpha, & (\alpha\beta)^n \\ (\beta\alpha)^n\beta, & (\beta\alpha)^n \end{array} \mid n \in \mathbb{N} \right\}$$

is a complete list of representatives.

If $\alpha = R_0$, $\beta = R_{\frac{1}{2}}$ then $R_{\frac{1}{2}}R_0 \colon x \to -x \to 1-(-x) = 1+x = T_1$.
So $\beta\alpha = T_1$, $\alpha\beta = T_{-1}$ is the inverse and $\mathbb{Z} < D_\infty$ is a subgroup of index two ie

$< \alpha\beta > << \alpha, \beta >$.

Remark 5.7. $T_1 R_0 = \beta\alpha\alpha = \beta = R_{\frac{1}{2}}$.

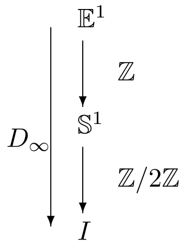


Figure 5.3: Taking quotients of $\mathbb{E}^1$ by discrete subgroups of $Isom(\mathbb{E}^1)$.

Exercise 5.8. : Classify discrete subgroups of $O(2) = \mathbb{S}^1 \rtimes \mathbb{Z}/2\mathbb{Z}$.