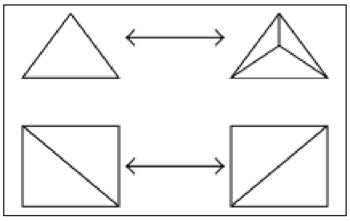
Saul Schleimer
Italo Cipriano

7 Lecture 7

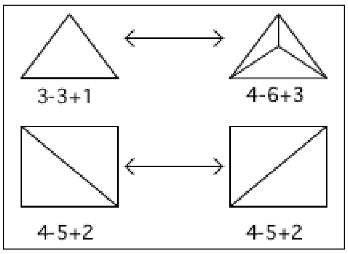
7.1 Aside from support class

A Pachner move (or bistellar flip) in dimension two is one of the following local moves:



Theorem 7.1 (Pachner)**.** *Any two triangulations of a closed surface are connected by a sequence of Pachner moves.*

Example 7.2 (An application of Pachner's theorem)**.** $\chi(F)$ is well defined: Prove this by showing that a single bistellar flip does not change Euler characteristic.
Check:



Example 7.3. As another application, show that orientability is independent of the choice of triangulation.

Exercise 7.4. Classify elements of $\text{Isom}(\mathbb{E}^2)$ as either the identity, a reflection in some line, a rotation about a point, a translation or a glide reflection.

Exercise 7.5. Do the same for $\text{Isom}(\mathbb{S}^2)$.

Exercise 7.6. Classify the discrete subgroups of $\text{Isom}(\mathbb{S}^1)$ [$\cong O(2)$].

7.2 Last time

Let X be one of $\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2$.

Theorem 7.7. *If F is modelled on X then there is a discrete subgroup $G < Isom(X)$, acting freely on X such that $F \cong X/G$ [Isometric].*

The last theorem says that if

$$A \doteq \{ \text{ Surface modelled on X} \} / \text{Isometry and}$$
$$B \doteq \{ \ G < Isom(X) \text{discrete, acting freely} \} / \text{Conjugation,}$$

then we have

$$B \ni G \mapsto X/G \in A \text{ and}$$
$$A \ni F \mapsto \text{Deck}(X \to F) \in B.$$

A major innovation of Thurston (Seifert, and others) is the drop of the hypothesis of freeness.

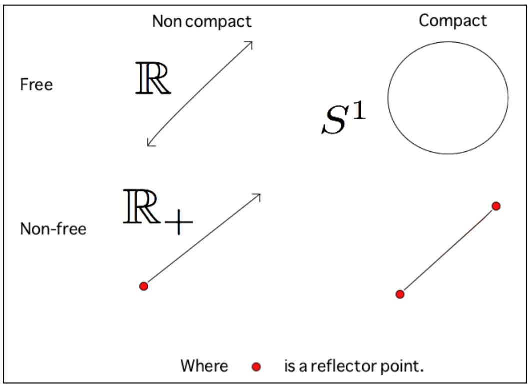
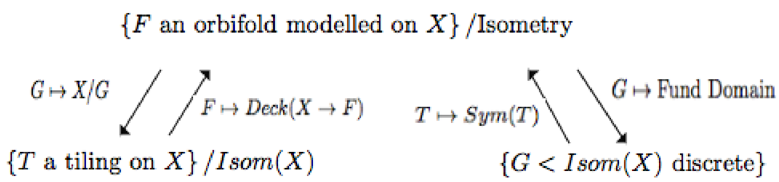


Figure 1: Discrete Quotients of $\mathbb{R}$.

The manifolds and orbifolds shown in Figure 1 correspond to the four isomorphism classes of discrete subgroup, $\mathbb{1}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}$ and D_∞, where

$$D_\infty \cong \langle \alpha, \beta \mid \alpha^2, \beta^2 \rangle .$$

If we abandon freeness we have the following diagram:



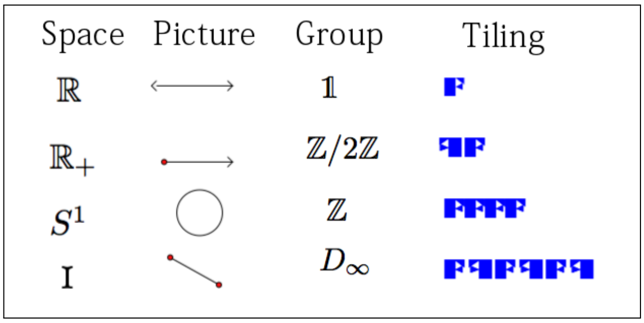


Figure 2: The discrete subgroups of $\mathrm{Isom}(\mathbb{E}^1)$.

Now we will go up one dimension. The *friezes* are planar tilings that repeat along a line. There are seven of them.

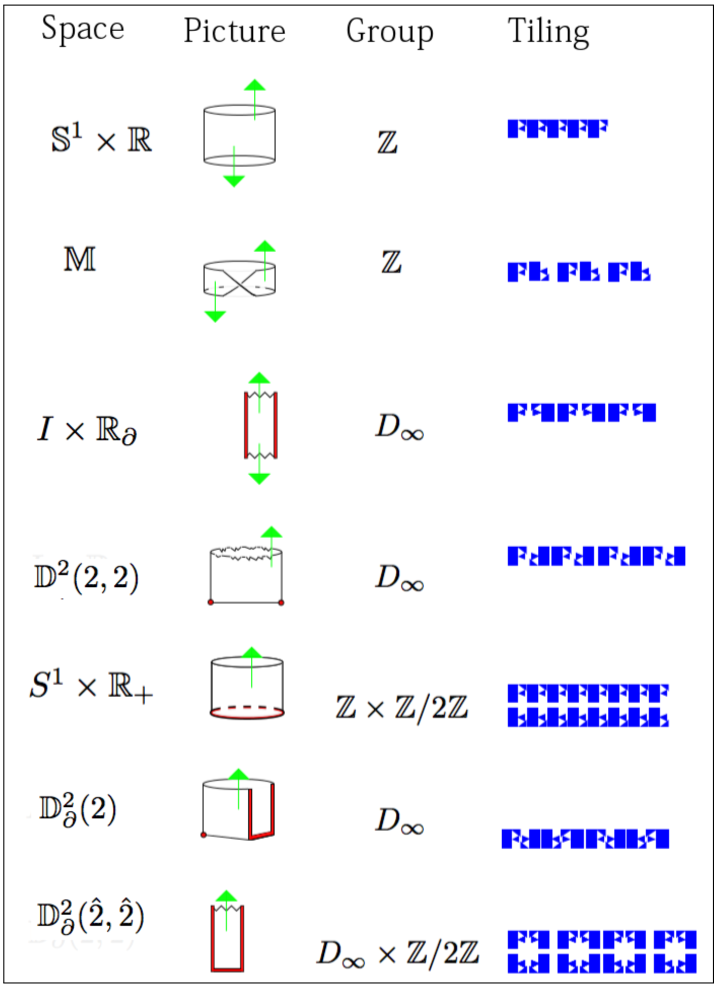


Figure 3: The seven frieze groups.