Saul Schleimer MA4J2
David Kaye 30th Jan 2011

9 Lecture 9

G is a *wallpaper group* if $G < \text{Isom}(\mathbb{E}^2)$, and G is discrete and acts cocompactly.

Example 9.1. $\mathbb{Z}^2$ with the usual identity is a wallpaper group. This is cocompact because

$$\mathbb{T}^2 = \mathbb{E}^2/\mathbb{Z}^2$$

is compact.

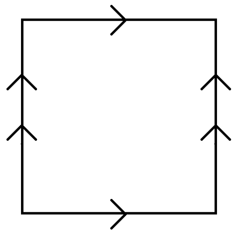

Figure 1: $\mathbb{T}^2$

Example 9.2. Likewise if $G = \langle g_1, g_2 \rangle$ with $g_1(x, y) = (x, y + 1)$ and $g_2(x, y) = (x + 1, -y)$, then

$$\mathbb{K} = \mathbb{E}^2/G$$

is compact. so G is a wallpaper group.

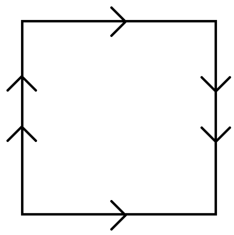

Figure 2: $\mathbb{K}$

These are the only wallpapers with quotient a manifold.

Theorem 9.3. *If $G < \text{Isom}(X)$ is discrete and acts freeely then $X/G = F$ is a manifold.*

Exercise 9.4. There are 15 more "types" of wallpaper groups, see [Scott] for a list.

If $G < \text{Isom}(X)$ is discrete then we say that $F = X/G$ is an *orbifold.*

Example 9.5. Let $G = \langle \alpha, \beta, \gamma \rangle$ where $\alpha, \beta, \gamma \in \text{Isom}(\mathbb{E}^2)$ are rotations by π about points P, Q, R (not all on a line) respectively. Then G is discrete (**Exercise**) and the quotient $\mathbb{E}^2/G$ is homeomorphic to a 2-sphere with four *cone points* of order two.

Notation: $S^2(2, 2, 2, 2) \cong \mathbb{E}^2/G$

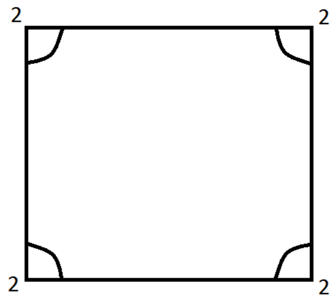

Figure 3: $S^2(2, 2, 2, 2)$

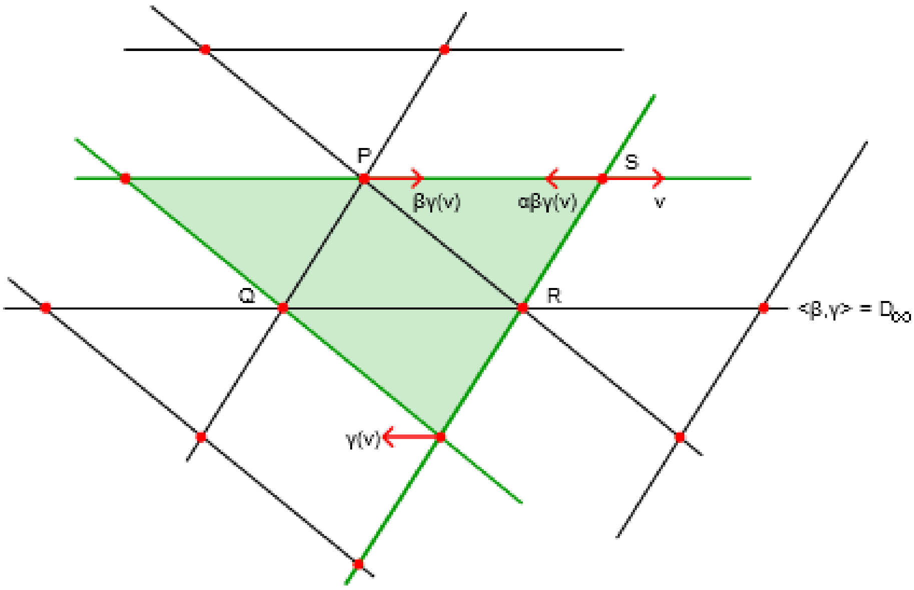

Figure 4: Action of G on $\mathbb{E}^2$

Remark 9.6. If $P = (0, 1)$, $Q = (0, 0)$, $R = (1, 0)$ then this orbifold is called the square pillowcase.

We may fold up the green triangle in Figure 4 (copied from lecture eight notes by George Frost) along the dotted lines as shown in Figure 5.

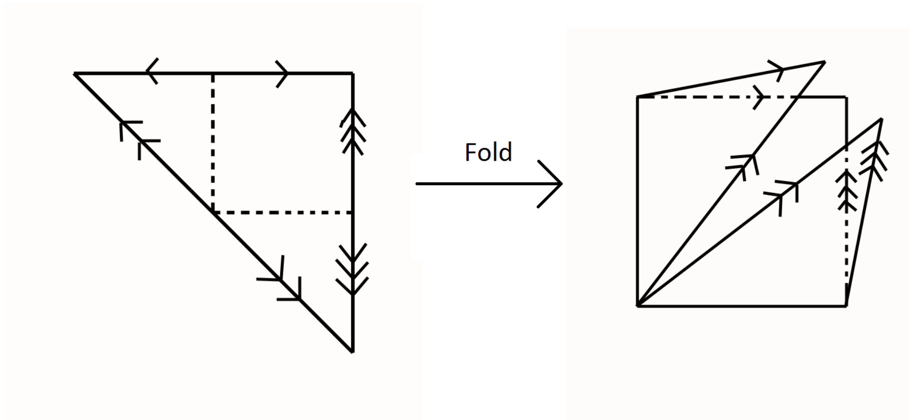

Figure 5: Construction of $S^2(2,2,2,2)$

Aside: Figure 6 is a sphere with cone points of angle $\frac{3\pi}{2}$; Hence this is *not* an orbifold.

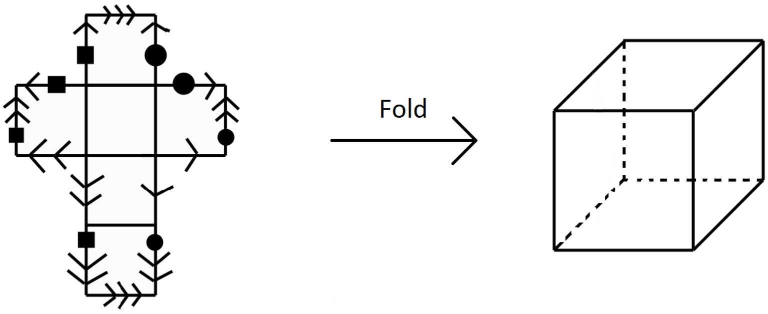

Figure 6: Construction of a cube

9.1 Local Pictures

For a 2-orbifold, all points have a neighbourhood U, homeomorphic to one of the following local pictures:

$$U \subseteq \mathbb{R}^2/\text{finite group} \quad \text{or} \quad U \subseteq \mathbb{R}^2_+/\text{finite group}$$

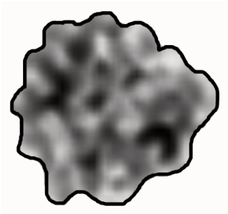

Figure 7: $U \subseteq \mathbb{R}^2/\langle Id \rangle$

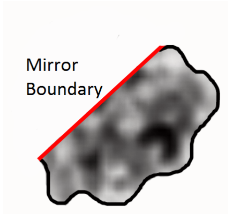

Figure 8: $U \subseteq \mathbb{R}^2/\langle \text{reflection} \rangle$

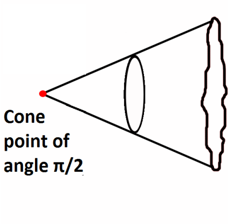

Figure 9: $U \subseteq \mathbb{R}^2/C_n$

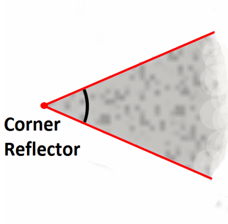

Figure 10: $U \subseteq \mathbb{R}^2/D_{2n}$

Figure 11: $U \subseteq \mathbb{R}^2_+$

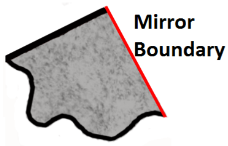

Figure 12: $U \subseteq \mathbb{R}^2_+/\langle \text{reflection} \rangle$

Definition 9.7. A 2-orbifold is a (Hausdorff, second-countable) topological space, F such that every $x \in F$ has a neighbourhood $U \subseteq F$ homeomorphic to one of the above local pictures. See [Scott] for a more general definition (i.e. for general dimension).

That is: F is a surface together with a finite collection of *cone points*, *corner reflectors* (with their orders) and a set of arcs in the boundary that are called *mirrors*.

The cube is S^2 together with eight "cone points" of angle $\frac{3\pi}{2}$. This is not an orbifold because $\frac{3\pi}{2} \neq \frac{2\pi}{n}$ for any $n = 1, 2, 3, \ldots$

9.2 Tetrahedron (of side length 1)

This is a ($\mathbb{E}^2$) orbifold. We can show this by counting the total angle at the four cone points. In fact this is $S^2(2, 2, 2, 2)$ again.

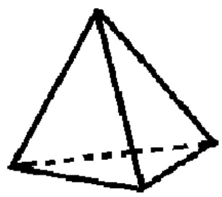

Figure 13: $S^2(2,2,2,2)$