MA3H60

THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: MAY 2012

ALGEBRAIC TOPOLOGY

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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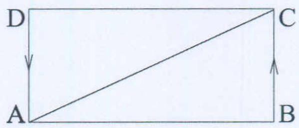
1. (a) The following diagram shows a rectangle whose two ends, BC and DA, are identified to make a Möbius strip M. Use the diagram to give M a Δ-complex structure, and compute the boundary of each of the simplices in this structure. [4]



(b) Using your answer to (a), choose bases for $\Delta_i(M)$ for $i = 0, 1, 2$ and exhibit the chain complex $\Delta_\bullet(M)$ as a sequence of free abelian groups and explicit matrices. [4]

(c) Use this complex to calculate the homology groups of M, taking care to give explicit definitions of any isomorphisms you use. [6]

(d) Now let N be the boundary of the Möbius strip. By using some of the simplices of the Δ-complex structure you gave to M, give N a Δ-complex structure. Compute $H_1(N)$, and determine, as a homomorphism of abstract groups, the map $H_1(N) \to H_1(M)$ that results from the inclusion of chain complexes $\Delta_\bullet(N) \hookrightarrow \Delta_\bullet(M)$. [4]

(e) Using the calculations you have done, calculate the relative homology groups $H_1(M, N)$ and $H_2(M, N)$, as abstract groups. Indicate also simplicial chain(s) generating $H_1(M, N)$. [4]

(f) Describe the space M/N. [3]

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2. Suppose that $0 \longrightarrow A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \longrightarrow 0$ is a short exact sequence of chain complexes. In lectures we showed how to define the connecting homomorphism $\partial : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ and proved exactness of the sequence

$$\cdots \longrightarrow H_{n+1}(C_\bullet) \xrightarrow{\partial} H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{j_*} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \longrightarrow \cdots.$$

To do:

(a) Explain the construction of ∂, and show that it is well defined. **[12]**

(b) Show how to construct the long exact sequence of *reduced* homology of a pair (X, A) **[3]**

(c) The *Snake Lemma* says that given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \end{array}$$

in which the rows are exact, there is an exact sequence

$$0 \to \mathrm{Ker}(\varphi_1) \to \mathrm{Ker}(\varphi_2) \to \mathrm{Ker}(\varphi_3) \to \mathrm{Coker}(\varphi_1) \to \mathrm{Coker}(\varphi_2) \to \mathrm{Coker}(\varphi_3) \to 0.$$

Show that this is a special case of the long exact sequence coming from a short exact sequence of complexes. **[3]**

(d) Let $C_n(X; \mathbb{Z}_n)$ be the group of chains with coefficients in $\mathbb{Z}_n$ instead of in $\mathbb{Z}$, and let $H_k(X; \mathbb{Z}_n)$ denote the homology of the chain complex $C_\bullet(X; \mathbb{Z}_n)$. Show that there is a short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet(X) \xrightarrow{\times n} C_\bullet(X) \longrightarrow C_\bullet(X; \mathbb{Z}_n) \longrightarrow 0$$

where $\times n$ is the map which multiplies all the coefficients in a chain by n. Deduce that there is a short exact sequence

$$0 \longrightarrow H_k(X)/nH_k(X) \longrightarrow H_k(X; \mathbb{Z}_n) \longrightarrow n\text{-torsion}(H_{k-1}(X)) \longrightarrow 0$$

where for an Abelian group A, n-torsion(A) denotes the kernel of the multiplication map $A \xrightarrow{n} A$. **[7]**

continued

3. (a) State the excision theorem. [3]

(b) Let X be a CW complex, and, for each $n \in \mathbb{N}$, let X^n be its n-skeleton.

(i) Show that $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$, and prove that $H_n(X^n, X^{n-1})$ is isomorphic to the free abelian group on the n-cells of X. [4]

(ii) Use the long exact sequence of the pair (X^n, X^{n-1}) to show that $H_k(X^n) = 0$ if $k > n$. [4]

(iii) Show, with the help of a diagram, how to construct the boundary maps

$$H_n(X^n, X^{n-1}) \rightarrow H_{n-1}(X^{n-1}, X^{n-2})$$

in the Cellular Chain Complex, and show that they make it into a complex. Be sure to leave enough space for your diagram! [4]

(iv) Show that the homology of the cellular chain complex is isomorphic to ordinary (singular) homology. For this you may assume without proof that the inclusion $X^n \hookrightarrow X$ induces isomorphisms $H_k(X^n) \rightarrow H^k(X)$ if $k < n$. [5]

(v) State the Cellular Boundary Formula, and use it to compute the homology of the space obtained from $S^1 \vee S^1$ by gluing in two 2-cells by the words $abab$ and $a^3ba{-}1$. [5]

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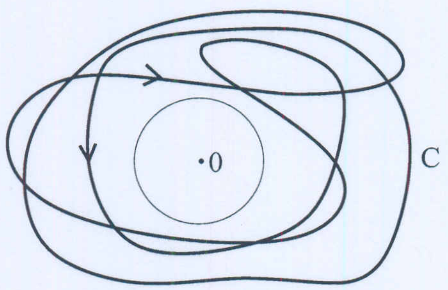
4. (a) Define the *degree* of a continuous map $S^n \longrightarrow S^n$. [3]

(b) Suppose that $f : S^n \longrightarrow S^n$, and that for some point $y_0 \in S^n$, $f^{-1}(y_0) = \{x_1, \ldots, x_m\}$. Define the *local degree* of f at x_i, and show how to calculate it when $n = 1$. [6]

(c) The thick line in the following diagram shows the image C of a map $f : S^1 \longrightarrow \mathbb{R}^2$, with the arrows indicating the direction of travel as the parameter $\theta \in S^1$ increases from 0 to 2π.



Define a map $g : S^1 \longrightarrow S^1$ by composing f with radial projection to S^1: $g = r \circ f$, where $r(y) = y/\|y\|$. What is the degree of g? [5]

(d) Let $f : S^n \longrightarrow S^n$ be a map of degree 0. Show that there exist points $x_1, x_2 \in S^n$ such that $f(x_1) = x_1$ and $f(x_2) = -x_2$. [4]

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(e) Suppose that V is a continuous vector field on the unit $(n+1)$-ball D^{n+1} such that $V(x) \neq 0$ for all x. Define a map $f : S^n \to S^n$ by

$$f(x) = \frac{V(x)}{||V(x)||}.$$

Show that there must be a point on S^n where V points radially outwards, and a point on ∂D^{n+1} where V points radially inwards. [7]

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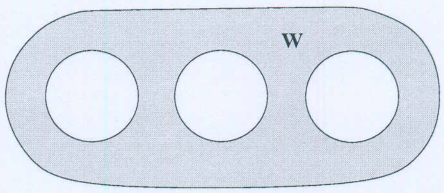
5. (a) The following drawing shows a region W in the plane. What are the homology groups $H_k(W)$ for $k = 0, 1$ and 2? Draw a picture showing closed curves whose homology classes form a basis for $H_1(W)$. No explanation is required here.



[3]

(b) Show in a picture the surface X that is obtained when two copies W_1 and W_2 of W are glued together by the identity map of their boundary. On your picture, draw closed curves whose homology classes form a basis for $H_1(X)$. No justification is required here. [4]

(c) Explain briefly what is the *Mayer-Vietoris* exact sequence and where it comes from, describing in particular the construction of the connecting homomorphism. [8]

(d) By slightly fattening W_1 and W_2 in the surface X of (c), we can assume they are both open in X. Using your prior knowledge of the homology of X, write down the Mayer Vietoris sequence associated to the two open sets W_1 and W_2, using the bases for the groups H_1 you have given in (a) and in (b), introducing further bases where necessary, and showing matrices of the homomorphisms in the sequence. Give brief explanations. [10]

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