Please let me know if any of the problems are unclear or have typos.

Exercise 4.1. [Hatcher page 147, the splitting lemma.] Suppose that $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is a short exact sequence of Abelian groups. Show that the following are equivalent.

- There is a homomorphism $r\colon B \to A$ so that $ri = \mathrm{Id}_A$.
- There is a homomorphism $s\colon C \to B$ so that $ps = \mathrm{Id}_C$.
- $B \cong A \oplus C$ and there is an isomorphism from the given sequence to the sequence $0 \to A \to A \oplus C \to C \to 0$ with the obvious inclusion and projection maps.

Such a sequence is called *split*; the maps r and s are called a *retraction* and a *section*, respectively.

Exercise 4.2. [Hatcher page 148.] With notation as in Exercise 4.1, show that if C is free then the sequence is split.

Exercise 4.3. We say a exact sequence is *very short* if it has at most two non-trivial terms.

i. Show that, when non-trivial, the central map of a very short exact sequence is an isomorphism.

ii. Show that a long exact sequence of free Abelian groups may be written as a direct sum of very short exact sequences.

Exercise 4.4. [Roberts.] For each exact sequence of Abelian groups below, say as much as possible about the group G and the homomorphism α.

i. $0 \to \mathbb{Z}_2 \to G \to \mathbb{Z} \to 0$

ii. $0 \to \mathbb{Z} \to G \to \mathbb{Z}_3 \to 0$

iii. $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2 \to 0$

iv. $0 \to G \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}_2 \to 0$

v. $0 \to \mathbb{Z}_3 \to G \to \mathbb{Z}_2 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$

Here we write $\mathbb{Z}_n$ for the group $\mathbb{Z}/n\mathbb{Z}$.

Exercise 4.5. [Medium.] Suppose that B and D are finitely generated free Abelian groups, $A < B$ and $C < D$ are subgroups, and $B/A \cong D/C$. Show that the chain complexes $0 \to A \to B \to 0$ and $0 \to C \to D \to 0$ are chain homotopy equivalent. (This is a generalization of Exercise 3.10. Smith normal form may be useful. See also problem 43(b) on page 159 of Hatcher.)

Exercise 4.6. [Hard.] With definitions as in Exercise 3.10, show that chain complexes C_* and D_* of finitely generated free Abelian groups are chain homotopy equivalent if and only if they have isomorphic homology groups: $H_*(C_*) \cong H_*(D_*)$. (Hints are available at MathOverflow, question number 10974. Exercise 4.5 may be useful. See also problem 43(a) on page 159 of Hatcher.)

Exercise 4.7. [Medium.] Find two topological spaces X and Y, with isomorphic homology groups, that are not homotopy equivalent. (Thus Exercise 4.6 does not generalize to topological spaces.) State any theorems from Hatcher that you use; you will need to read ahead a bit.