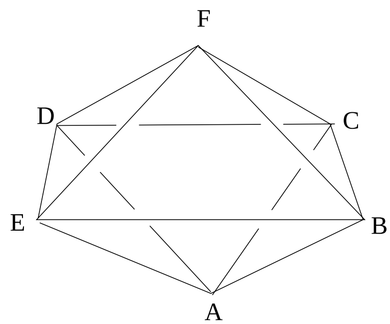
Algebraic Topology   Assessed Exercises I

To be handed in by midday on Friday of week 6

1. Construct a Δ-complex structure on the Klein bottle K, and use it to compute the simplicial homology groups of K.

2. (i) The diagram below shows a polyhedron X homomorphic to the 2-sphere S^2. Define an equivalence relation $\sim$ on X which corresponds to the antipodal identification on S^2.
(ii) Construct a Δ-complex structure on S^2 using eight 2-simplices as shown, in such a way that



it passes to the quotient by $\sim$ to give a Δ-complex structure on $\mathbb{R}\mathbb{P}^2$.
(iii) Write down the matrix of each of the maps $\partial_j : \Delta_j(\mathbb{R}\mathbb{P}^2) \to \Delta_{j-1}(\mathbb{R}\mathbb{P}^2)$ in the chain complex resulting from (i), with respect to suitable bases.
(iv) Compute $H_1(\mathbb{R}\mathbb{P}^2)$ using the chain complex of (iii). The answer should be that $H_1(\mathbb{R}\mathbb{P}^2) \simeq \mathbb{Z}/2\mathbb{Z}$, of course, as it was when we calculated in it lectures using a Δ-complex structure with two 2-simplices (though we have not yet proved that the homology groups are independent of the choice of Δ-complex structure we choose). In any case, you should go carefully through the details in this calculation and reach the correct answer honestly!

3. Show that there are no retractions $r : X \to A$ in the following cases:

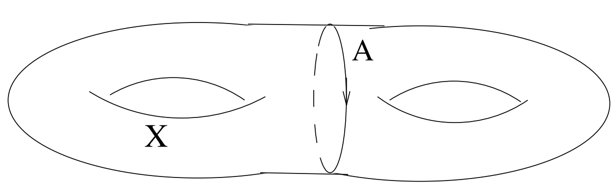
1. $X = \mathbb{R}^3$ and A any subspace homeomorphic to S^1.
2. $X = S^1 \times D^2$ and A its boundary torus $S^1 \times S^1$.
3. $X = D^2 \vee D^2$ and A its boundary $S^1 \vee S^1$.
4. X the Möbius band and A its bounding circle.

4.(i) Show that if $B \xrightarrow{f} \mathbb{Z}^n$ is an epimorphism, with B abelian, then there is a homomorphism $g : \mathbb{Z}^n \to B$ such that $f \circ g = \mathbf{1}_{\mathbb{Z}^n}$. Hint: all you need to do is choose the value of g on the generators $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ of $\mathbb{Z}^n$. Then g extends to a homomorphism $\mathbb{Z}^n \to B$ "by linearity".
(ii) Deduce that if

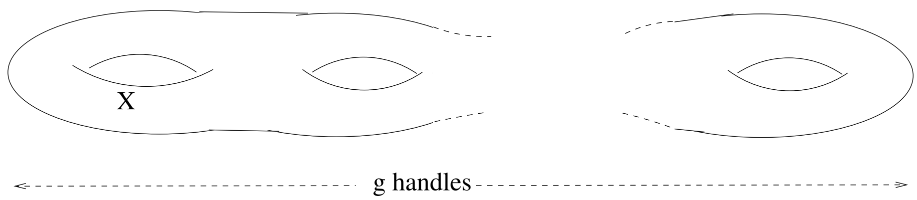
$$0 \to A \to B \xrightarrow{f} \mathbb{Z}^n \to 0$$

is a short exact sequence of Abelian groups then $B \simeq A \oplus \mathbb{Z}^n$. Hint: taking g as in (i), $\mathbf{1} - g \circ f$ maps B into $\ker f$.

5. Let X and A be the genus 2 surface and the circle shown in the following diagram.



Show by a drawing that A, suitably divided into 1-simplices, is the boundary of a 2-chain in X. Conclude that the map $H_1(A) \to H_1(X)$ induced by the inclusion $A \hookrightarrow X$ is zero. Hint: your drawing might be easier to make using the representation of the torus as the quotient of a square.
(ii) Compute $H_1(X, A)$, and go on to compute $H_1(X)$.
(iii) Compute the relative homology group $H_2(X, A)$
(iv) Generalise the argument of (ii) to compute $H_1(X)$, where X is the genus g surface.



6. Show that if X is a genus g surface with k small circular holes cut in it, then $H_1(X)$ is isomorphic to $\mathbb{Z}^n$ for some n, and determine the value of n. Hint: if A is the boundary of one of the holes, what is X/A?

7. Can there be an exact sequence

$$0 \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to 0 \quad ?$$

8.(i) Suppose that $0 \longrightarrow A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \longrightarrow 0$ is a short exact sequence of chain complexes. In lectures we showed how to define the connecting homomorphism $\partial : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ and proved exactness of the sequence

$$\cdots \longrightarrow H_{n+1}(C_\bullet) \xrightarrow{\partial} H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{j_*} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \longrightarrow \cdots$$

at $H_n(B_\bullet)$. Prove exactness at $H_n(A_\bullet)$.
(ii) The *snake lemma* says that a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow p & & \downarrow q & & \downarrow r & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \end{array}$$

in which the rows are exact gives rise to an exact sequence

$$0 \longrightarrow \ker(p) \longrightarrow \ker(q) \longrightarrow \ker(r) \longrightarrow \operatorname{coker}(p) \longrightarrow \operatorname{coker}(q) \longrightarrow \operatorname{coker}(r) \longrightarrow 0.$$

Show that this is a special case of the long exact sequence arising from a short exact sequence of complexes, as described in (ii).

9. (i) Show that $X := S^1 \times S^1$ and $Y := S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions.
(ii) (harder?) Does there exist a continuous map $X \to Y$ or $Y \to X$ simultaneously inducing isomorphisms in all homology groups?