The required problems are Exercises 7.1, 7.2, and 7.3. Please let me know if any of the problems are unclear or have typos.

Exercise 7.1. Give a purely algebraic proof of the key lemma: if $P, Q \in \mathbb{H}^2$ are points in the hyperbolic plane then $P \circ Q \leq -1$, with equality if and only if $P = Q$. [Hint: Apply the Cauchy-Schwarz inequality to the second and third coordinates.]

Exercise 7.2. Suppose that $u \in \mathbb{L}^3$ is a vector in lorentzian space. Define the *Lorentz orthogonal* to be $u^\perp = \{v \in \mathbb{L}^3 \mid u \circ v = 0\}$.

- Show that $u^\perp \subset \mathbb{L}^3$ is a two-dimensional linear subspace for any nonzero $u \in \mathbb{L}^3$.
- Show that if u is time-like then every non-zero $v \in u^\perp$ is space-like.

Exercise 7.3. Fix distinct points $P, Q \in \mathbb{H}^2$. Set $C = d_H(P, Q)$. We define

$$P' = \frac{P - \cosh(C)Q}{\sinh(C)} \qquad \text{and} \qquad L(t) = \cosh(t)Q + \sinh(t)P'.$$

- Verify that $P' \circ Q = 0$, $P' \circ P' = 1$, and $P' \circ P > 0$.
- Show that $L(t) \in \mathbb{H}^2$, for all t.
- Show that $L(0) = Q$ and $L(C) = P$.
- More generally, show $d_H(Q, L(t)) = |t|$, for all t.

We deduce that $L(t)$ is a great hyperbola, parametrised by hyperbolic distance.

Exercise 7.4. Suppose that $t \geq 0$ and $\theta \in [0, 2\pi)$. Define

$$P(t, \theta) = (\cosh(t), \sinh(t)\cos(\theta), \sinh(t)\sin(\theta)).$$

This defines *polar coordinates* on $\mathbb{H}^2$.

- Show that $P(t, \theta) \in \mathbb{H}^2$, for all t, θ.
- Show that, for any $Q \in \mathbb{H}^2$, there are coordinates t, θ so that $Q = P(t, \theta)$.
- Give an explicit basis for $P(t, \theta)^\perp$.
- Set $O = (1, 0, 0)$. Prove $d_H(O, P(t, \theta)) = t$.
- Set $O = (1, 0, 0)$. Suppose that $\theta, \sigma \in [0, \pi)$. Show that the angle $\Theta(P, O, R)$ between $P = P(t, \theta)$ and $R = P(s, \sigma)$ is $|\theta - \sigma|$.