Please let me (Saul) know if any of the problems are unclear or have typos. Please turn in a solution to one of Exercises 6.1, 6.3, or 6.4 by 14:00 on 2017-11-16 in front of the undergraduate office. If you collaborate with other students, please include their names.

Exercise 6.1. Recall that $T^2 = S^1 \times S^1$ is the two-torus; informally T^2 is the surface of a donut. Fix any point $x \in T^2$; show that $T^2 - \{x\}$ deformation retracts to the figure-eight graph. Illustrate your proof with useful figures.

Exercise 6.2. [A version of Exercise 14, page 39, of Hatcher's book.] Given topological spaces X and Y we equip $Z = X \times Y$ with the product topology. Let $p\colon X \times Y \to X$ be projection to the first factor; that is $p(a, b) = a$. Define $q\colon X \times Y \to Y$ to be projection to the second factor.

Fix $x \in X$ and $y \in Y$. Prove that the homomorphism

$$p_* \times q_* \colon \pi_1(X \times Y, (x, y)) \to \pi_1(X, x) \times \pi_1(Y, y)$$

is an isomorphism. (Essentially you are being asked to carefully reprove Proposition 1.12, using the notion of projections.)

Exercise 6.3. The *real projective space* $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. We equip $\mathbb{RP}^n$ with its usual topology, coming from the round metric; the distance between distinct lines $L, M \subset \mathbb{R}^{n+1}$ is the smaller of the two angles made by L and M in the plane they span.

- Exhibit a two-fold covering map $p\colon S^n \to \mathbb{RP}^n$.
- Deduce that $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$, when $n \geq 2$.
- Briefly discuss the cases of $n = 0$ and $n = 1$. Give pictures.

Exercise 6.4. Suppose that $p\colon \widetilde{X} \to X$ is a d–fold covering map and that $\widetilde{X}$ is path-connected. Prove that $\mathrm{Deck}(p)$ has at most d elements. Give examples which do and which do not realize this bound.

Exercise 6.5. [Hard.] Let X and Y be copies of the two-sphere and choose distinct points $p, p' \in X$ and $q, q' \in Y$. Define

$$Z = {}^{X \sqcup Y}\big/_{p \sim q,\ p' \sim q'}$$

to be the quotient space. That is, Z is obtained from the disjoint union of X and Y by identifying p with q and p' with q'. Draw a picture of Z. Compute $\pi_1(Z)$ and carefully justify your reasoning.