Please let me (Saul) know if any of the problems are unclear or have typos. Please turn a solution to one of Exercise 3.2, Exercise 3.3, or Exercise 3.10 by 14:00 Wednesday after next, in front of the undergraduate office. If you collaborate with other students, please include their names.

Exercise 3.1. Compute the simplicial homology groups of the two-sphere, directly from the definitions, using the Δ–complex structure coming from the boundary of a tetrahedron. Provide clearly labelled figures.

Exercise 3.2. Suppose that X is a non-empty path-connected space, equipped with a Δ–complex structure. Show, directly from the definitions, that $H_0^\Delta(X) \cong \mathbb{Z}$. (You may assume without proof that the one-skeleton is path-connected.)

Exercise 3.3. Suppose that X is a finite, path-connected, one-dimensional Δ–complex: that is, a finite connected graph. Suppose that X has E edges and V vertices. Compute the simplicial homology groups of X.

Exercise 3.4. Compute the reduced singular homology groups of a point, directly from the definitions.

Exercise 3.5. [Challenge.] Compute the singular homology groups of the circle S^1, directly from the definitions.

Exercise 3.6. Compute the singular homology groups of the plane $\mathbb{R}^2$, minus n points. Reference any theorems from Hatcher that you use.

Exercise 3.7. Suppose $f_\#\colon \mathcal{C}_* \to \mathcal{D}_*$ is a *chain map* of chain complexes: a sequence of group homomorphisms $\{f_n\}$ so that $\partial_n^D \circ f_n = f_{n-1} \circ \partial_n^C$. The short-hand for this is $\partial f_\# = f_\# \partial$. Show $f_\#$ induces well-defined homomorphisms $f_*\colon H_*(\mathcal{C}) \to H_*(\mathcal{D})$ on homology.

Exercise 3.8.

- Suppose that X is a Δ–complex. Let $i_k\colon \mathcal{C}_k^\Delta(X) \to \mathcal{C}_k^s(X)$ be the injective homomorphisms sending a simplex of the Δ–complex to the corresponding singular simplex. Show that $i_\# = \{i_k\}$ is a chain map. (Later in the course we will prove $i_\#$ induces an isomorphism from simplicial homology to singular.)
- Suppose that $f\colon X \to Y$ is a map of topological spaces. For any singular n–simplex $\sigma\colon \Delta^n \to X$ define $f_n(\sigma) = f \circ \sigma$. Extend linearly to get a homomorphism $f_n\colon C_n^s(X) \to C_n^s(Y)$. Show that $f_\# = \{f_n\}$ is a chain map.

Exercise 3.9. [Hatcher problem 12, page 132.] Two chain maps $f_\#$ and $g_\#$ from $\mathcal{C}_*$ to $\mathcal{D}_*$ are *chain homotopic* if there is a sequence of homomorphisms $P_n\colon C_n \to D_{n+1}$ so that

$$\partial_{n+1}^D \circ P_n + P_{n-1} \circ \partial_n^C = g_n - f_n$$

for all n. The short-hand for this is $\partial P + P\partial = g_\# - f_\#$. We call P a *chain homotopy* and write $f_\# \sim g_\#$. Show chain homotopy of chain maps is an equivalence relation.

Exercise 3.10. We say two chain complexes $\mathcal{C}_*$ and $\mathcal{D}_*$ are *chain homotopy equivalent* if there are chain maps $f_\#\colon \mathcal{C}_* \to \mathcal{D}_*$ and $g_\#\colon \mathcal{D}_* \to \mathcal{C}_*$ so that $g_\# \circ f_\# \sim \mathrm{Id}_C$ and $f_\# \circ g_\# \sim \mathrm{Id}_D$.

Let $\mathcal{C}_*$ be the chain complex with $C_1 = C_0 = \mathbb{Z}$, all other chain groups trivial, and with $\partial_1^C(m) = 2m$. Let $\mathcal{D}_*$ be the chain complex with $D_1 = D_0 = \mathbb{Z}^2$, all other chain groups trivial, and with $\partial_1^D(x, y) = (x - y, x + y)$. Prove $\mathcal{C}_*$ and $\mathcal{D}_*$ are chain homotopy equivalent.