Please let me (Saul) know if any of the problems are unclear or have typos. Please turn a solution to one of Exercise 4.1, Exercise 4.5, or Exercise 4.6 by 14:00 Wednesday after next, in front of the undergraduate office. If you collaborate with other students, please include their names.

Exercise 4.1. For each of the following spaces draw a Δ–complex structure with at most a pair of two-simplices: B^2 the disk, S^2 the sphere, P^2 the real projective plane, C^2 the cylinder, M^2 the Möbius band, T^2 the torus, and K^2 the Klein bottle. Also, state their reduced homology groups (do not show your computations).

Exercise 4.2. [Hatcher page 147, the splitting lemma.] Suppose that $\mathcal{C}_*$ is a chain complex. If $H_*(\mathcal{C}) = 0$ we call $\mathcal{C}_*$ an *exact sequence*. (Equivalently, $Z_n = B_n$ for all n.) If, additionally, $\mathcal{C}_*$ has at most three non-zero terms we call $\mathcal{C}_*$ a *short exact sequence*. Show that every non-zero term is adjacent to another such. Now suppose

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is a short exact sequence of abelian groups. Show the following are equivalent.

- There is a homomorphism $r\colon B \to A$ so that $ri = \mathrm{Id}_A$.
- There is a homomorphism $s\colon C \to B$ so that $ps = \mathrm{Id}_C$.
- $B \cong A \oplus C$ and there is an isomorphism from the given sequence to the sequence $0 \to A \to A \oplus C \to C \to 0$ with the obvious inclusion and projection maps.

Such a sequence is called *split*; the maps r and s are called a *retraction* and a *section*, respectively.

Exercise 4.3. [Hatcher page 148.] With notation as in Exercise 4.2, show if C is free then the sequence is split.

Exercise 4.4.

1. For any $d \times n$ matrix M of integers the *Smith normal form* of M is a $d \times n$ matrix $D = [d_{i,j}]$ with $d_{i,j} \in \mathbb{N}$, with $d_{i,j} = 0$ (if $i \neq j$), with $d_{i,i}$ evenly dividing $d_{i+1,i+1}$, and having matrices $U \in \mathrm{GL}(d, \mathbb{Z})$ and $V \in \mathrm{GL}(n, \mathbb{Z})$ so that $D = UMV$. Learn how to compute Smith normal form in your favourite computer algebra system.
2. Find an algorithm that, given any chain complex $\mathcal{C}_* = \{C_n, \partial_n\}$ of finitely generated free abelian groups, computes the homology groups $H_*(\mathcal{C})$. (Hints: Recall $B_n = \mathrm{Im}(\partial_{n+1})$ and $Z_n = \mathrm{Ker}(\partial_n)$. Inclusion (of Z_n into C_n) and ∂_n give a short exact sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$. Since B_{n-1} is free, this sequence splits. Choose a splitting $C_n \cong Z_n \oplus B_{n-1}$ extending the inclusion of Z_n into C_n. This induces an isomorphism $\mathrm{Coker}(\partial_{n+1}) \cong H_n \oplus B_{n-1}$.)

3. Show that the following sequence of groups and homomorphisms $\mathcal{C}_*$ is a chain complex.

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{M} \mathbb{Z}^5 \xrightarrow{N} \mathbb{Z} \longrightarrow 0$$

Here the homomorphisms M and N are given by the matrices

$$M = \begin{bmatrix} -6 & -26 & -82 \\ 0 & 4 & 3 \\ 1 & 0 & 7 \\ 0 & 2 & 5 \\ 2 & 10 & 30 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} -2 & -2 & -4 & -2 & -4 \end{bmatrix}.$$

4. Take M, N, and $\mathcal{C}_*$ as in part (3) above. Compute the Smith normal forms of M and N. Use the algorithm to compute $H_*(\mathcal{C})$.

Exercise 4.5. We say an exact sequence is *very short* if it has at most two non-zero terms. Suppose that $\mathcal{C}_*$ is a non-trivial very short exact sequence.

i. Show that the non-zero terms of $\mathcal{C}_*$ are adjacent.

ii. Show the central map of $\mathcal{C}_*$ is an isomorphism.

iii. Show that an exact sequence $\mathcal{D}_*$ of free abelian groups may be written as a direct sum of very short exact sequences.

Exercise 4.6. [Roberts.] For each exact sequence of abelian groups below, say as much as possible about the group G and the homomorphism α.

i. $0 \to \mathbb{Z}_2 \to G \to \mathbb{Z} \to 0$

ii. $0 \to \mathbb{Z} \to G \to \mathbb{Z}_3 \to 0$

iii. $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2 \to 0$

iv. $0 \to G \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}_2 \to 0$

v. $0 \to \mathbb{Z}_3 \to G \to \mathbb{Z}_2 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$

Here we write $\mathbb{Z}_n$ for the group $\mathbb{Z}/n\mathbb{Z}$.

Exercise 4.7. [Medium.] Suppose B and D are finitely generated free abelian groups, $A < B$ and $C < D$ are subgroups, and $B/A \cong D/C$. Show the chain complexes $0 \to A \to B \to 0$ and $0 \to C \to D \to 0$ are chain homotopy equivalent. (This is a generalization of Exercise 3.10. Smith normal form may be useful. See also problem 43(b) on page 159 of Hatcher.)

Exercise 4.8. [Hard.] Show chain complexes $\mathcal{C}_*$ and $\mathcal{D}_*$ of finitely generated free abelian groups are chain homotopy equivalent if and only if they have isomorphic homology groups: $H_*(\mathcal{C}) \cong H_*(\mathcal{D})$. (Hints are available at MathOverflow, question number 10974. Exercise 4.7 may be useful. See also problem 43(a) on page 159 of Hatcher.)

Exercise 4.9. [Medium.] Find two topological spaces X and Y, with isomorphic homology groups, that are not homotopy equivalent. (Thus Exercise 4.8 does not generalize to topological spaces.) State any theorems from Hatcher that you use; you will need to read ahead a bit.