

Please let me (Saul) know if any of the problems are unclear or have typos. Please turn a solution to one of Exercise 6.1, Exercise 6.2, or Exercise 6.3 by 14:00 on 2017-02-22, in front of the undergraduate office. If you collaborate with other students, please include their names.

Exercise 6.1.

- Let X be a copy of the Möbius band. Let $A = \partial X$ be the topological boundary of X . Find a Δ -complex structure on X and thus on A .
- Let X be a copy of the two-dimensional torus, minus a small open disk. Let $A = \partial X$ be the topological boundary of X . Find a Δ -complex structure on X and thus on A .

In both cases, compute the simplicial homology groups of A and of X , the homomorphisms induced by inclusion, and the relative homology groups of the pair (X, A) .

Exercise 6.2.

- Show if (X, A) is a good pair then so is $(X/A, A/A)$. [Harder] What about the converse?
- Suppose H is the Hawaiian earring and 0 is the origin. Show $(H, \{0\})$ is not a good pair.

Exercise 6.3. Prove the inclusions $(B^n, S^{n-1}) \subset (B^n, B^n - \{0\}) \subset (\mathbb{R}^n, \mathbb{R}^n - \{0\})$ induce isomorphisms of relative homologies. (This is used in the proof of invariance of domain.)

Exercise 6.4. [Medium. See Hatcher page 133, problem 27, for definitions.] Show the inclusion $i: (B^n, S^{n-1}) \rightarrow (B^n, B^n - \{0\})$ is not a homotopy equivalence of pairs. (This rules out one possible approach to Exercise 6.3.)

Exercise 6.5. Suppose $h: B^n \rightarrow B^n$ is a fixed-point-free map. For any $x \in B^n$, let L_x be the straight line through $h(x)$ and x oriented from $h(x)$ towards x . For points $y, z \in L_x$ we write $y < z$ if z comes after y along L_x . Let $f(x)$ and $g(x)$ be the two points of $L_x \cap S^{n-1}$ ordered to ensure $g(x) \leq h(x) < x \leq f(x)$. Prove $f: B^n \rightarrow S^{n-1}$ is well-defined, continuous, and a retraction. (This is a step of the proof of the Brouwer fixed point theorem.)

Exercise 6.6. [Hard. Hatcher page 133, problems 20 and 21.] Define CX , the *cone* of X , to be $X \times I / X \times \{1\}$. Define SX , the *suspension* of X , to be the space obtained by doubling CX across its base. That is, SX is obtained from $X \times [-1, 1]$ by crushing $X \times \{-1\}$ and $X \times \{1\}$ to points.

- Prove SS^n is homeomorphic to S^{n+1} .
- Find chain maps $s_n: C_n(X) \rightarrow C_{n+1}(SX)$ that induce isomorphisms $s_{n*}: H_n(X) \rightarrow H_{n+1}(SX)$.