Please let me (Saul) know if any of the problems are unclear or have typos. Please turn a solution to one of Exercise 9.1, Exercise 9.4, or Exercise 9.7 by 14:00 on 2017-03-15, in front of the undergraduate office. If you collaborate with other students, please include their names.

Exercise 9.1. "Let $p(z) = z^3$ and $q(z) = z^2$; these are polynomials defined on the complex plane. Let $f(z,t) = (1-t)p(z) + tq(z)$, for $t \in [0,1]$. Thus f is a homotopy from p to q. Let P, Q, and F be the extensions of p, q, and f to the one-point compactification $S^2 = \mathbb{C} \cup \{\infty\}$. Thus $\deg(P) = 3$ and $\deg(Q) = 2$. Also, F is a homotopy from P to Q. Deduce $\deg(P) = \deg(Q)$, and so $3 = 2$."

Verify the true statements and disprove the false ones.

Exercise 9.2. Fix $\{a_k\}_{k=0}^d \subset \mathbb{R}$ with $a_d \neq 0$. Define $p \colon \mathbb{R}^1 \to \mathbb{R}^1$ by $p(x) = \sum a_k x^k$. Let P be the extension of p to the one-point compactification $S^1 = \mathbb{R}^1 \cup \{\infty\}$. Check P is continuous. Compute the degree of P.

Exercise 9.3. [Hatcher page 137, Propostion 2.33.] Recalling the definitions from Exercise 6.6, if $f \colon X \to X$ is a map then define $Sf \colon SX \to SX$ to be the *suspension* of f: that is, the self-map of SX induced by $f \times \mathrm{Id}$. Prove, when X is a sphere, that $\deg(f) = \deg(Sf)$.

Exercise 9.4. We say a map $f \colon S^n \to S^n$ is *even* if $f(-x) = f(x)$ for all $x \in S^n$. Prove if $f \colon S^n \to S^n$ is even then $\deg(f)$ is even. (You may restrict to the cases where $n = 1$ and $n = 2$.)

Exercise 9.5. [Borsuk. Hard.] We call a map g *odd* if for all x we have $g(-x) = -g(x)$.

Show that the following are equivalent.

1. For any map $f \colon S^n \to \mathbb{R}^n$ there is a point $x \in S^n$ so that $f(-x) = f(x)$.
2. For any odd map $g \colon S^n \to \mathbb{R}^n$ there is a point $x \in S^n$ so that $g(x) = 0$.
3. There is no odd map $h \colon S^n \to S^{n-1}$.
4. There is no map $k \colon B^n \to S^{n-1}$ so that $k|S^{n-1}$ is odd.

Now deduce one (and so all) of these from the "odd theorem": an odd map $f \colon S^n \to S^n$ has odd degree.

Exercise 9.6. Set $C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$. For all points $p \in C$ compute the local homology groups of C at p.

Exercise 9.7. Recalling the definitions from Exercise 6.6, show that the cone $C(\mathbb{RP}^2)$ is not a three-manifold with boundary. (You may use without proof the fact that $C(Z)$ is contractible, for any space Z.)