Please send me (Saul) any corrections and/or improvements to the exercises or their solutions.

Exercise 1.7. Suppose that $\Delta = \Delta^n = [e_0, e_1, \ldots, e_n]$ is the standard simplex sitting in $\mathbb{R}^{n+1}$. Count its k–faces.

Solution. Note that Δ has $n+1$ vertices. Any k–face F of Δ is a k–simplex and so has $k+1$ vertices. Thus to determine the face F we must choose $k+1$ of the e_i. There are $\binom{n+1}{k+1}$ ways to do this. $\square$

Exercise 1.8. Let $X = \mathbb{R}/\mathbb{Q}$ be the topological quotient: that is, $x \sim y$ if $x, y \in \mathbb{Q}$ or if $x = y$. Find all open sets in X.

Solution. We take $\pi\colon \mathbb{R} \to X$ to be the quotient map; so $\pi(x) = [x]$ is the equivalence class of x. Recall that in the quotient topology $U \subset X$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}$. If U is empty, then $\pi^{-1}(U)$ is empty, and so is open in $\mathbb{R}$. Thus the empty set is open in X.

Suppose instead that U is an open, non-empty set in X. Thus $V = \pi^{-1}(U)$ is open and non-empty in $\mathbb{R}$. Thus V is a non-trivial union of open intervals. Thus V contains some rational number r. Thus $[r]$ lies in U. Since $[r] = \mathbb{Q}$, we deduce that $\mathbb{Q} \subset V$. Thus V is open and dense. Thus $V = \mathbb{R}$ and so $U = X$.

It follows that X has exactly two open sets: the empty set and X itself. $\square$