Please send me (Saul) any corrections and/or improvements to the exercises or their solutions.

Exercise 2.8. [Hard.] Compute the simplicial homology groups of Δ^n, the n–simplex equipped with the natural Δ–complex structure.

Later in the module, once we have developed more machinery, we will be able to give a "simple" proof. Nonetheless, there are some direct approaches. Here is one relying on the fact that the simplex is a *cone*.

Solution of Exercise 2.8. Set $X = \Delta^n = [e_0, e_1, \ldots, e_n]$. We claim that $H_0^\Delta(X) \cong \mathbb{Z}$ and $H_k^\Delta(X) \cong 0$ if $k > 0$. That is, Δ^n has the same homology groups as a point.

Let $I \subset \{0, 1, \ldots, n\}$ be a subset. If I is non-empty we define $\sigma_I = [e_i]_{i \in I}$. If $0 \in I$ then we call σ_I an *upper* face of X. (That is, if $e_0 \in \sigma_I$.) If $0 \notin I$ then we call σ_I a *lower* face.

We now define I', as follows.

$$I' = \begin{cases} I - \{0\}, & \text{if } 0 \in I \\ I \cup \{0\}, & \text{if } 0 \notin I \end{cases}$$

If neither I nor I' are empty then we say that σ_I and $\sigma_{I'}$ are *paired*. Note that $[e_0]$ is the only simplex not paired with some other simplex.

Suppose that $\sigma_{I'}$ is upper. Then σ_I is lower and is the first term in the sum given by $\partial \sigma_{I'}$. Furthermore, σ_I is the only lower face appearing in the sum. Finally, $\sigma_{I'}$ is the only upper simplex having σ_I in its boundary.

Let $C_k^{\text{up}}, C_k^{\text{low}} \subset C_k^\Delta(X)$ be the subgroups generated by the upper and lower k–faces, respectively. Since every simplex is either upper or lower, but not both, we deduce $C_k^\Delta(X) = C_k^{\text{up}} \oplus C_k^{\text{low}}$. Let $\beta_k \colon C_k^\Delta(X) \to C_k^{\text{low}}$ be the associated projection. Note that applying $\beta_k \circ \partial_{k+1}$ to an upper face gives the paired lower face. Since the pairing is a bijection (for $k > 0$) we deduce that $\beta_k \circ \partial_{k+1} | C_{k+1}^{\text{up}}$ is an isomorphism, for all $k > 0$.

Claim. For all k we have $\partial_k(C_k^{\text{low}}) \subset \partial_k(C_k^{\text{up}})$.

Proof. Suppose that σ_I is a lower k–face. We must prove that $\partial_k \sigma_I \in \partial_k(C_k^{\text{up}})$. By the remarks above (in the fourth paragraph of the proof), $\partial_{k+1}\sigma_{I'} = \sigma_I + c$ for some $c \in C_k^{\text{up}}$. By Lemma 2.1 [Hatcher] we have $\partial_k \partial_{k+1} \sigma_{I'} = 0$. Thus $\partial_k \sigma_I = -\partial_k c$. □

We deduce that $C_k^\Delta = C_k^{\text{up}} \oplus \text{Im}(\partial_{k+1})$. Since $\partial_k | C_k^{\text{up}}$ is injective, we deduce that $\text{Ker}(\partial_k)$ is equal to $\text{Im}(\partial_{k+1})$, as long as $k > 0$. For $k = 0$, there is exactly one zero-face not paired with an upper one-face, namely $[e_0]$. We deduce that $H_0^\Delta(X) \cong \mathbb{Z}$ while $H_k^\Delta(X) \cong 0$ for $k > 0$. □

Exercise 2.10. Let $X = \mathbb{RP}^2$. Give a Δ–complex structure on X. Now compute the simplicial homology groups $H_*^\Delta(X)$.

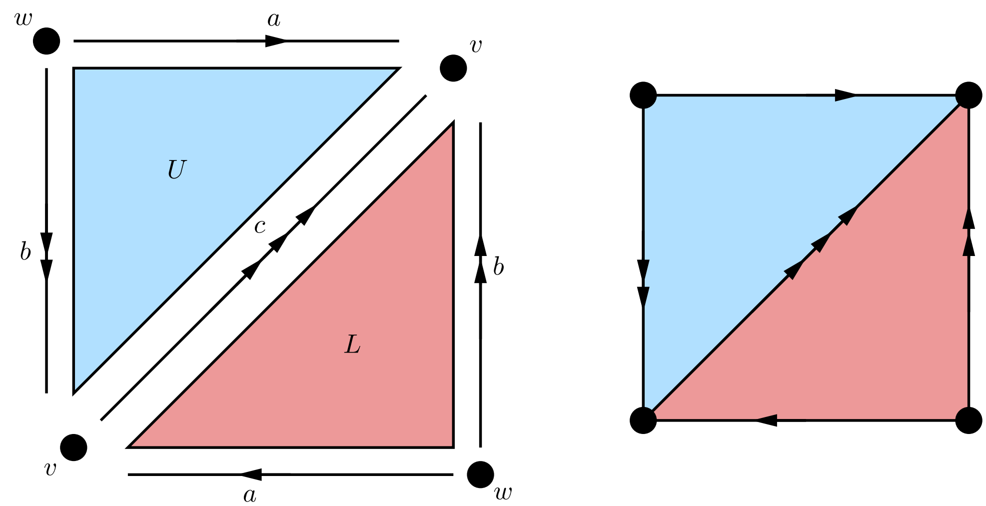

Figure 2.11: Left: The exploded Δ–complex structure for $\mathbb{RP}^2$ with the one-simplices v, w, a, and b shown twice. Right: The identification space. On both sides, the arrows indicate both orientation and gluings.

Solution. See Figure 2.11 (right) for picture of the Δ–complex structure on X. The simplices are denoted by U, L (triangles), a, b, c (edges), and v, w (vertices). Note that the orientations on the edges of a Δ–complex always point *from* the smaller vertex and *towards* the larger. In this example, if we reverse the direction of the edge a then we get one of the usual identification spaces giving $\mathbb{RP}^2$; thus X is homeomorphic to $\mathbb{RP}^2$. Note, however, that that "usual" orientation of a prevents the cell-complex from having a Δ–complex structure.

In what follows, if A is an abelian group and $B \subset A$ is subset, then we use $\langle B\rangle$ to denote the subgroup of A generated by B.

The above Δ–complex structure on X has the following chain groups.

$$C_2^\Delta(X) = \langle U, L\rangle \cong \mathbb{Z}^2$$
$$C_1^\Delta(X) = \langle a, b, c\rangle \cong \mathbb{Z}^3$$
$$C_0^\Delta(X) = \langle v, w\rangle \cong \mathbb{Z}^2$$

We now compute the boundary of every simplex σ, using the orientation of the edges to deduce the order of the vertices in σ.

$$\partial_2 U = c - a + b = -a + b + c \qquad \partial_1 a = v - w \qquad \partial_0 v = 0$$
$$\partial_2 L = c - b + a = a - b + c \qquad \partial_1 b = v - w \qquad \partial_0 w = 0$$
$$\partial_1 c = v - v = 0$$

Note that $\mathrm{Ker}(\partial_0) = \langle v, w\rangle = C_0^\Delta(X)$. We change basis to arrange $\langle v, w\rangle = \langle v - w, w\rangle \cong \mathbb{Z}^2$. On the other hand $\mathrm{Im}(\partial_1) = \langle v - w\rangle \cong \mathbb{Z}$. Thus

$$H_0^\Delta(X) = \frac{\langle v - w, w\rangle}{\langle v - w\rangle} \cong \mathbb{Z}.$$

Note that $\mathrm{Ker}(\partial_1) = \langle a - b, c\rangle$. [To prove this, suppose that p, q, and r are any integers and suppose that $d = pa + qb + rc$ is a one-cycle. Thus $\partial d = (p + q)(v - w) = 0$ and so $q = -p$. Thus $d = p(a-b)+rc$.] We change basis to arrange $\mathrm{Ker}(\partial_1) = \langle a - b + c, c\rangle \cong \mathbb{Z}^2$. On the other hand we may change basis to arrange $\mathrm{Im}(\partial_2) = \langle -a + b + c, a - b + c\rangle = \langle 2c, a - b + c\rangle = \langle a - b + c, 2c\rangle \cong \mathbb{Z}^2$. Thus

$$H_1^\Delta(X) = \frac{\langle a - b + c, c\rangle}{\langle a - b + c, 2c\rangle} \cong \mathbb{Z}/2\mathbb{Z}.$$

Note that $\mathrm{Ker}(\partial_2) = 0$. [To prove this, suppose that p and q are any integers and suppose that $D = pU + qL$ is a two-cycle. Then $\partial D = p(-a + b + c) + q(a - b + c) = (-p + q)a + (p - q)b + (p + q)c = 0$. Thus $p - q = 0$ and $p + q = 0$. The only solution in integers is $p = q = 0$. Thus $D = 0$, as desired.] Thus

$$H_2^\Delta(X) = 0$$

and we are done. □

The computations in Exercise 2.10 amount to finding matrix forms for each boundary operator, putting these matrices in Smith normal form, and tracking the change of bases that occur. For a very readable discussion of how to compute homology groups using Smith normal form, see Section 17.6 of [`http://jeffe.cs.illinois.edu/teaching/comptop/2009/notes/homology.pdf`].