Please send me (Saul) any corrections and/or improvements to the exercises or their solutions.

Exercise 3.11. Suppose that C_* and D_* are chain complexes. We define their *direct sum* to be the chain complex $E_* = C_* \oplus D_*$ with groups $E_k = C_k \oplus D_k$ and with boundary homomorphisms $\partial_k^E = \partial_k^C \oplus \partial_k^D$. Show that $H_*(E) = H_*(C) \oplus H_*(D)$.

Solution. We have $Z_*(E) = \mathrm{Ker}(\partial_*^E) = \mathrm{Ker}(\partial_*^C) \oplus \mathrm{Ker}(\partial_*^D) = Z_*(C) \oplus Z_*(D)$. Likewise we have $B_*(E) = \mathrm{Im}(\partial_*^E) = \mathrm{Im}(\partial_*^C) \oplus \mathrm{Im}(\partial_*^D) = B_*(C) \oplus B_*(D)$. We may now compute

$$H_*(E) = \frac{Z_*(E)}{B_*(E)} = \frac{Z_*(C) \oplus Z_*(D)}{B_*(C) \oplus B_*(D)} = H_*(C) \oplus H_*(D)$$

as desired. $\square$

Exercise 3.12. Suppose that C is the chain complex with $C_0 \cong C_1 \cong \mathbb{Z}^3$, all other groups zero, and ∂_1 given by the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Decompose C as the sum of four non-trivial chain complexes.

Solution. We find the Smith normal form of ∂_1; it is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus there are bases for C_1 and for C_0 so that ∂_1 has this matrix form. We deduce that C decomposes as the sum of the following chain complexes:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1} \mathbb{Z} \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z} \xrightarrow{3} \mathbb{Z} \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

The last of these further decomposes as the sum of the following chain complexes:

$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow 0$$
$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

The vertical alignment indicates the index of each group in each complex. $\square$