Please send me (Saul) any corrections and/or improvements to the exercises or their solutions.

**Exercise 5.10.** Compute the reduced simplicial homology groups of $\Delta^n$, the $n$–simplex equipped with the natural $\Delta$–complex structure.

This is a slight variant on Exercise 2.8. Using a chain homotopy we can give a more "conceptual" proof.

*Solution of Exercise 5.10.* Set $X = \Delta^n = [e_0, e_1, \ldots, e_n]$. To show that $H_*^\Delta(X) \cong 0$ it suffices to prove that the augmented chain complex $C_* = \widetilde{C}_*^\Delta(X)$ is exact.

Let $I \subset \{0, 1, \ldots, n\}$ be a subset. We define $\sigma_I = [e_i]_{i \in I}$, where the order of the vertices comes from the order of the indices. If $I$ is empty, then $\sigma_I$ is the generator of $C_{-1} \cong \mathbb{Z}$. We define $I'$ as follows.

$$I' = \begin{cases} I - \{0\}, & \text{if } 0 \in I \\ I \cup \{0\}, & \text{if } 0 \notin I \end{cases}$$

Note that $I'' = I$. We define $P\colon C_k \to C_{k+1}$ by taking

$$P(\sigma_I) = \begin{cases} 0, & \text{if } 0 \in I \\ \sigma_{I'}, & \text{if } 0 \notin I \end{cases}$$

and extending linearly.

*Claim.* $P$ is a chain homotopy from the identity to the zero map. That is, $\partial P + P\partial = \mathbb{1}$.

*Proof.* It suffices to check this on a basis element $\sigma_I$. Suppose that $i_0 < i_1 < \ldots < i_k$ are the elements of $I$. There are two cases: either 0 lies in $I$ or it does not. Suppose that $0 \in I$. We compute:

$$\begin{aligned} (\partial P + P\partial)\sigma_I &= P\partial\sigma_I && [P(\sigma_I) = 0] \\ &= P\big(\sigma_{I'} + \sum_{j>0} (-1)^j \sigma_{I-\{i_j\}}\big) && [\text{definition of } \partial] \\ &= \sigma_I + \sum_{j>0} (-1)^j P(\sigma_{I-\{i_j\}}) && [P(\sigma_{I'}) = \sigma_I] \\ &= \sigma_I && [\text{definition of } P] \end{aligned}$$

Now suppose that $0 \notin I$. We compute:

$$\begin{aligned} (\partial P + P\partial)\sigma_I &= \partial\sigma_{I'} + P\partial\sigma_I && [P(\sigma_I) = \sigma_{I'}] \\ &= \sigma_I + \sum_{j \geq 0} (-1)^{j+1} \sigma_{I'-\{i_j\}} + P\big(\sum_{j \geq 0} (-1)^j \sigma_{I-\{i_j\}}\big) && [\text{definition of } \partial] \\ &= \sigma_I + \sum_{j \geq 0} (-1)^{j+1} \sigma_{I'-\{i_j\}} + \sum_{j \geq 0} (-1)^j \sigma_{I'-\{i_j\}} && [\text{definition of } P] \\ &= \sigma_I && [\text{cancellation}] \end{aligned}$$

This proves the claim. □

Now suppose that $z \in Z_k$ is a cycle. Define $c = Pz$. We compute:

$$\begin{aligned} \partial c &= \partial P z && \text{[definition of } c\text{]} \\ &= z - P\partial z && [\partial P + P\partial = \mathbb{1}] \\ &= z && [z \text{ is a cycle}] \end{aligned}$$

Thus $z$ is a boundary, the chain complex $C_*$ is exact, and we are done. □

**Exercise 5.11.** Compute the reduced simplicial homology groups of $S^{n-1}$, using the $\Delta$-complex structure coming from the homeomorphism $S^{n-1} \cong \partial\Delta^n$.

*Solution.* Define

$$A_* = \widetilde{C}_*^\Delta(\partial\Delta^n), \qquad B_* = \widetilde{C}_*^\Delta(\Delta^n), \qquad C_* = C_*^\Delta(\Delta^n, \partial\Delta^n).$$

We are asked to compute the homology groups of $A_*$. From the definition of $C_*$ we know that

$$0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0$$

is a short exact sequence. Thus there is an exact triangle of homology groups. In Exercise 5.10 we showed that $H_*(B_*) = 0$. Exactness of the triangle now implies that $H_k(C_*) \cong H_{k-1}(A_*)$, for all $k$.

Note that the complex $C_*$ has exactly one non-zero term, namely $C_n \cong \mathbb{Z}$. Thus $H_k(C_*) = \mathbb{Z}$ if $k = n$ and is zero otherwise. We deduce that $H_k(A_*) = \mathbb{Z}$ if $k = n - 1$ and is zero otherwise. □