Please send me (Saul) any corrections and/or improvements to the exercises or their solutions.

Exercise 6.7. Let X be a copy of the Klein bottle, minus a small open disk. Let $A = \partial X$ be the topological boundary of X.

- Find a Δ-complex structure on X and thus on A.
- Using this, compute the simplical homology groups of A and of X. Find the homomorphism on homology induced by the inclusion of A into X.
- Compute the relative homology groups of the pair (X, A).

This is a slight variant on Exercise 6.1.

Solution of Exercise 6.7. We first build a Δ–complex structure for X.

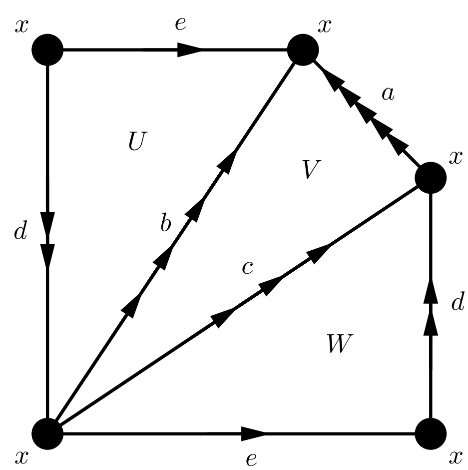

Figure 6.8: The Δ–complex structure for X has three two-simplices U, V, W, has five one-simplices a, b, c, d, e, and has one zero-simplex x. The arrows indicate both orientation and glueings.

Note that A is the subcomplex containing the zero-simplex x and the one-simplex a. We obtain the following nontrivial chain groups.

$$C_2^\Delta(X) = \langle U, V, W\rangle \cong \mathbb{Z}^3 \qquad C_1^\Delta(A) = \langle a\rangle \cong \mathbb{Z}$$
$$C_1^\Delta(X) = \langle a, b, c, d, e\rangle \cong \mathbb{Z}^5 \qquad C_0^\Delta(A) = \langle x\rangle \cong \mathbb{Z}$$
$$C_0^\Delta(X) = \langle x\rangle \cong \mathbb{Z}$$

We now record the image of the boundary operator acting on each simplex.

$$\begin{aligned} & & \partial_1 a &= 0 & & \\ \partial_2 U &= d + b - e & \partial_1 b &= 0 & & \\ \partial_2 V &= c + a - b & \partial_1 c &= 0 & \partial_0 x &= 0 \\ \partial_2 W &= e + d - c & \partial_1 d &= 0 & & \\ & & \partial_1 e &= 0 & & \end{aligned}$$

Since the Δ-complex structure on A is the usual structure on the circle, we find that $H_1^\Delta(A) = \langle [a] \rangle \cong \mathbb{Z}$ and $H_0^\Delta(A) = \langle [x] \rangle \cong \mathbb{Z}$.

Note that $\mathrm{Ker}(\partial_0) = \langle x \rangle = C_0^\Delta(X)$ while $\mathrm{Im}(\partial_1) = 0$. Thus $H_0^\Delta(X) = \langle [x] \rangle \cong \mathbb{Z}$. The induced homomorphism $H_0^\Delta(A) \to H_0^\Delta(X)$ is thus the identity, because both groups are generated by $[x]$.

Note that $\mathrm{Ker}(\partial_1) = C_1^\Delta(X)$. We change basis to obtain

$$\begin{aligned} \mathrm{Ker}(\partial_1) &= \langle a, b, c, d, e \rangle \\ &= \langle a, d + b - e, c, d, e \rangle \\ &= \langle a, d + b - e, e + d - c, d, e \rangle \\ &= \langle a + 2d, d + b - e, e + d - c, d, e \rangle \end{aligned}$$

We can also change basis in $C_2^\Delta(X)$ to obtain $\langle U, V, W \rangle = \langle U, U + V + W, W \rangle$ with image $\mathrm{Im}(\partial_2) = \langle a + 2d, d + b - e, e + d - c \rangle$. Thus $H_1^\Delta(X) = \langle [d], [e] \rangle \cong \mathbb{Z}^2$. Also, since $\mathrm{Im}(\partial_2)$ contains three elements of a basis, we deduce that ∂_2 is injective. So $\mathrm{Ker}(\partial_2) = 0 = H_2^\Delta(X)$. We also record the fact that, in $H_1^\Delta(X)$, we have $[a + 2d] = 0$, and so $[a] = -2[d]$.

The induced homomorphism $H_0^\Delta(A) \to H_0^\Delta(X)$ takes $[a]$ to $-2[d]$ and thus is injective.

We now turn to the computation of $H_*^\Delta(X, A)$. Note that $C_0^\Delta(X, A) \cong 0$ because the induced homomorphism on chains is surjective. Thus $H_0^\Delta(X, A) = 0$. On the other hand

$$\begin{aligned} C_1^\Delta(X, A) &= \langle [b], [c], [d], [e] \rangle \\ &= \langle [d + b - e], [c], [d], [e] \rangle \\ &= \langle [d + b - e], [e + d - c], [d], [e] \rangle \end{aligned}$$

Here the square brackets denote relative chains, not homology classes. Note that ∂_1 induces the zero homomorphism while ∂_2 induces a homomorphism with image $\langle [2d], [d + b - e], [e + d - c] \rangle$. Thus $H_1^\Delta(X, A) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ where the first factor is generated by $[d]$ and the second is generated by $[e]$. Finally, ∂_2 is injective on relative two-chains, so $H_2^\Delta(X, A) = 0$. □

Remark 6.9. As always, complicated computations in simplical homology are easier to carry out using Smith normal form. Also, it helps to know the answer before you set out. Since X deformation retracts to $d \cup e$ we are confident about our calculation of $H_*(X)$. Since (X, A) is a good pair, and since X/A is homeomorphic to the Klein bottle, we are likewise confident of our calculation of $H_1(X, A)$.