Intro to 3-mfds

Overview: Topology (mfds | maps (diffeo, homeo)) Geometry

I Basics

n-mfds = top space locally modelled on $\mathbb{R}^n$

with ∂ : $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$ upper half space

Fundamental problem

Given M, N mfds, is $M \cong N$?

n=0: Solution: Count

Now suppose connected

n=1: There are 4: $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $I = [0,1]$, $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$

Let's assume compact

n=2: A few examples to start

$S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\} \cong \mathbb{CP}^1 = \frac{\mathbb{C}^2 - 0}{(z,w) \sim (\lambda z, \lambda w)}$ for $\lambda \neq 0$

$T^2 = S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$

$A^2 = S^1 \times I$, the interior is $\cong \mathbb{R}^2/\mathbb{Z}$ ("from above (universal cover)", "from below (cell str)")

$D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$

$P^2 = \mathbb{RP}^2 = \frac{\mathbb{R}^3 - 0}{\mathbb{R}^* - 0} = \mathbb{P}(\mathbb{R}^3) \cong S^2/x \sim -x$ [$\mathbb{P}(V)$ = space of lines in V]

Def: Suppose M, N are n-mfds and $B, C \subset M, N$ are emb n-balls. Fix $\varphi : \partial B \to \partial C$ a homeo [all very nice]

Define $M \# N$ [the connect sum] to be $M \# N = \frac{(M - \mathrm{int}(B)) \sqcup (N - \mathrm{int}(C))}{x \sim \varphi(x)}$

Ex: $K \cong \mathbb{P}^2 \# \mathbb{P}^2 \sim 2\mathbb{P}^2$ [$k\mathbb{P}^2 = \#_{i=1}^{k} \mathbb{P}^2$]

Def: Suppose $\varphi : S^1 \to S^1$ a reflection. Define $S^1 \times_\varphi S^1 = \frac{S \times I}{(x,1) \sim (\varphi(x), 0)}$ φ $\cong$

Theorem: Every compact, connected surface is homeo to one of $S_{g,n,b} = S^2 \# g\mathbb{T}^2 \# c\mathbb{P}^2 \# b\mathring{D}^2$ (genus, crosscaps, boundaries)

Ex: ① $A^2 = 2D^2$ ② $T^2 \# \mathbb{P}^2 = 3\mathbb{P}^2$ (only relation)

[cf Conway's ZIP proof]

$S_{3,2,2}$

Thurston: Auto connect sum form $M \#_\varphi = \frac{M - \mathrm{int}(B \cup C)}{x \sim \varphi(x)}$ $B \cup C \subset M$ and eg $S^2 \#_\varphi \mathbb{T}^2 \cong \mathbb{P}^2$

Q: Is the orientation [reverse = reverse] all that matters? [check which φ does what]

A: It depends on $MCG(S^{n-1})$ and the category you are working in

Def: $MCG(M) = \frac{\mathrm{Homeo}(M)}{\mathrm{Homeo}_0(M)}$, $\mathrm{Homeo}(M) = \{\varphi \circ \mathrm{Homeo}(M) \mid \varphi \simeq \mathrm{Id}_M\}$ (isotopic)

We can ignore issues of category [Top, PL, Diff] because we are in dimensions ≤ 3

[cf Milnor's exotic 7-spheres, Taubes's exotic $\mathbb{R}^4$'s

Surgery Suppose $S \subset M^n$ is a nicely embedded S^{n-1}

So S has a product neighborhood $N(S) \cong S \times I$

Pick homeo $\varphi : \partial N \to \partial B^n \sqcup C$ [$B \subset B^n$]

and form $\dfrac{(M - n(S)) \sqcup B \times C}{x \sim \varphi(x)} = M_S$

M surgered along S

This is the inverse of connect sum

Q: So φ doesn't matter b/c B, C we disks? Q: why not?

A: Correct. A: Mfld homeo of S^{n-1} extend over B^n

$n=4$ The homeo problem is algorithmically unsolvable [But still very interesting]

Q: Why unsolvable [in one sentence]

A: In dim ≥ 4, presentations of $\pi_1(M)$ are unhelpful!

$n=3$ This module. Theme: $\pi_1(M)$ essentially controls the geometry of M and in an effective way.

In fact: Geometrization is effective [Algorithm] but not efficient

The difficulty of Homeo(3) is open [cf. G Kuperberg]

We typically assume M^3 is compact connected, oriented

(II) Elliptic Examples

$S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 2\}$

Fix $\zeta = \exp(\frac{2\pi i}{p})$ for $p \geq 2$ $p \in \mathbb{N}$

Fix also $q \in \mathbb{N}$. $1 \leq q \leq p-1$

Define $L(p,q) = S^3 / (z,w) \sim (\zeta z, \zeta^q w)$

The (p,q) lens space [cf. Reidemeister]

Rmk: It is possible to have $L(p,q) \cong L(p,q')$ but $L(p,q) \not\cong L(p,q')$

Ex $\pi_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$

Q: Why called lens space? Likewise: q matters to homeo type

A: From the following construction

U, L hemispheres of $S^2 = \partial B^3$

Ex $L(p,q) \cong B^3 / x \sim \varphi(x)$ $\varphi : U \to L$ $n = \frac{2\pi q}{p}$ rotation

Better picture

dihedral angle $= \pi/p$

(spherical lens)

Ex Show that this is geometrically correct [Hint: stereographic projection]

Brieskorn Manifolds

Define $T_p : S^1 \to S^1$, $(w \mapsto \zeta w)$

Rmk $S^1/\langle T_p \rangle = L(p,1)$

Define $J : S^1 \to S^1$, $(z,w) \mapsto (\bar{w}, \bar{z})$

Define $D_p^4 = \langle T_p, J \rangle$

Ex D_p^4 act freely on S^3. So $P_p = S^3/D_p^4$ is a 3-mfld

Ex Find a geometric cell decomposition [with one "point" 3-cell] and so justify the name

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Ex $\mathbb{RP}^3 = \mathbb{P}(\mathbb{R}^4) \cong L(2,1) \cong UTS^2$ [unit tangent]

Ex $S^3/\langle J \rangle \cong L(4,1) \cong UT(\mathbb{RP}^2)$

Ex Define boundary connect sum $\natural_\partial$ and so define handlebodies

$V_g = \natural_g S^1 \times D^2$

V_2