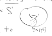
Lature 2

1. Please email me
   s.schleimer@warwick.ac.uk
2. Yair Minsky is speaking
   March 14,15 (for 2+2 hours)
   at Warwick

(III) Orientability

Def The solid torus is the manifold $\mathbb{D}^2 \times S^1$



Any disk (properly) isotopic to $D^2 \times \{pt\}$ is a meridional disk

$\left[\mathbb{D}^2 = \{ z \in \mathbb{C} \mid |z| \leq 1 \}\right]$

Def The solid Klein bottle is $\mathbb{D}^2 \times_\varphi S^1 \cong D^2 \tilde{\times} S^1$. This is hard to draw in $\mathbb{R}^3$

Here is $\mathbb{K}^2$ instead

That is,

$$SK = D^2 \tilde{\times} S^1 = \frac{D^2 \times I}{(z,1) \sim (\bar{z},0)}$$

There is a fibration

$$\mathbb{D}^2 \longrightarrow SK \longrightarrow S^1$$

The red is the core curve

Proposition

(1) A surface S is non-orientable iff there is some Möbius band $\mathbb{M}^2 \hookrightarrow S$

(2) A 3-mfd M is non-orientable iff there is some solid Klein bottle $SK \hookrightarrow M$

Pf. Depends on your def. of orientability $\square$

Q. Doesn't $\mathbb{M}^2$ embed in $S^1 \times \mathbb{D}^2$?

A. Yes. So orientable 3-mfds can contain interesting non-orientable surfaces.

I-bundles. Suppose F is a surface. Say B is an I-bundle over F if there is a fibration

$$I \longrightarrow B \longrightarrow F$$

(I.e. $\forall x \in F$ $\exists$ small neighborhood $x \in U \subseteq F$ st. $B|U \cong I \times U$)

Ex 1) $I \times F$ the product I-bund
2) $I \tilde{\times} F$ the orientation I-bund

For $B = I \tilde{\times} F$: $\alpha \hookrightarrow F$ is orient reversing iff $B|\alpha \cong \mathbb{M}^2$

Q. Are there other I-bundles?

A. Yes. Fix any homom $\varphi : \pi_1(F) \to \mathbb{Z}/2\mathbb{Z}$. Then $I \times_\varphi F$ is another I-bundle.

Ex. Only $I \times F$ and $I \tilde{\times} F$ are orientable as 3-manifolds

Ex. $SK \cong I \times_\varphi A^2$ $[\varphi : \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}]$

Ex. $S^1 \times \mathbb{D}^2 \cong I \tilde{\times} \mathbb{M}^2$

Rmk. If $B = I \times_\varphi F$ is an I-bund then there is a canonical section

$$\begin{array}{ccc} & (0,x) \longleftarrow x & \\ I \longrightarrow & B \longrightarrow & F \\ & (t,x) \longmapsto x & \end{array}$$

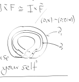
Q. Is $I \tilde{\times} F$ a quotient?

A. Yes, in many ways.

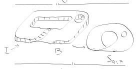
Eg. Let $\hat{F} \to F$ be the orient double cover. Let $\tau : \hat{F} \to \hat{F}$ be the involution $\hat{F}/\tau \cong F$

Check $[-1,1] \tilde{\times} F \cong I \times \hat{F} / (0,x) \sim (0,\tau(x))$

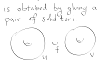
One dim lower



Ex. Draw these pictures yourself



Ex 1) Every lens space $L(p,q)$ is obtained by gluing a pair of solid tori.



$$f : \partial U \to \partial V$$

---

2) Every prism mfd is obtained by gluing $S^1 \times \mathbb{D}^2$ to $I \tilde{\times} \mathbb{K}^2$ (orient I-bund)

IV. Elliptic, Redux

Recall: $S^n = \{ x \in \mathbb{R}^{n+1} \mid |x| = 1 \}$

We equip S^n with the round metric: $d_S(u,v) = \arccos(u \cdot v)$

Ex: $\mathrm{Isom}(S^n) \cong O(n+1)$
$\mathrm{Isom}^+(S^n) \cong SO(n+1)$

Ex: Classify finite subgroups of $SO(3)$ and $SO(4)$ [Hard]

[cf Wolf: Spherical Space forms]

Actions: Please revise free proper actions. Also check that if $\Gamma : M \circlearrowleft$ is free, prop and M is mfd then M/Γ is again a manifold

Q: Proper versus prop discont

A: YES [Google Kapovich's discussion]

Ex: $\mathbb{RP}^3 \cong S^3/\langle -Id \rangle$

Exercise: If $\Gamma < O(4)$ and Γ acts free, prop on S^3 then S^3/Γ is orientable

---

Ex: Without using geometrization prove that if M is a 3-mfd and $\widetilde{M} \cong S^3$ [univ cover] then M is orientable

---

Now we turn to $S^2 \times \mathbb{R}$

We equip $S^2 \times \mathbb{R}$ with the product metric

Q: What is the canonical (const curv) metric on S^2?

A: The round metric [cf Gauss's Theorem]

In fact much less is required [cf Geometrization]

Ex: $\mathrm{Isom}(S^2 \times \mathbb{R}) \cong \mathrm{Isom}(S^2) \times \mathrm{Isom}(\mathbb{R})$
$\cong O(3) \times (\mathbb{R} \rtimes \mathbb{Z}_2)$

Ex: $\mathrm{Isom}(S^2 \times \mathbb{R})$ contains only seven families of subgroups acting freely and properly on $S^2 \times \mathbb{R}$. The resulting manifolds are

1. $S^2 \times \mathbb{R} \cong \mathbb{R}^3 - \{pt\} \cong S^3 - \{N,S\}$
2. $P^2 \times \mathbb{R}$
3. $\mathbb{R} \tilde{\times} \mathbb{P}^2 \cong \mathbb{RP}^3 - \{pt\}$
4. $S^2 \times S^1$
5. $S^2 \rtimes_f S^1 \cong S^2 \tilde{\times} S^1$
6. $P^2 \times S^1$
7. $S^1 \tilde{\times} P^2 \cong \mathbb{P}^3 \# \mathbb{P}^3$

Ex. Obtain the compact ones by gluing ST, SK, I-bund

(V) Irreducibility

Example: Alexander's horned sphere

This has a recursive construction

The limit S_∞ is an embedding of S^2 into S^3 but there is a Cantor set of points of the image which are not locally flat.

Ex: The outer region of $S^3 - S_\infty$ is not simply conn.

That is S_∞ is "wild"

Def: Suppose $F^2 \xhookrightarrow{f} M^3$ emb. A point $x \in F$ is locally flat if there is a neigh. $U \subseteq M$, $f(x) \in U$ st $(U, U \cap f(F)) \cong (\mathbb{R}^3, \mathbb{R}^2 \times \{0\})$

Alexander's Thm: Suppose $S^2 \xhookrightarrow{f} S^3$ is locally flat (or PL, or smooth). Then the closures of the components of $S^3 - f(S^2)$ are homeo to $\mathbb{B}^3$.

Def: M^3 is irreducible if every locally flat $S^2 \hookrightarrow M^3$ bounds a 3-ball.

Def: M^3 is prime if $M \cong N \# P$ implies $N \cong S^3$ or $P \cong S^3$

Ex: Find all prime, reducible mfds

Ex: Irred $\Rightarrow$ prime

Alexander's Thm: S^3 is irred.