Lecture 3

Business

1. Please use the email group to ask questions!
2. Yair Minsky now has a title for the talks on March 14-15 "The geometry and topology of Teichmuller space"

(VI) Alexander's Theorem

S^3 is irreducible

Recall.

Jordan-Schoenflies

Suppose $\alpha \subseteq S^2$ is a simple closed curve. Then the closures of the components of $S^2 - \alpha$ are disks.

In higher dim's some condition on the regularity of the embedding is needed. The smooth 4-dim'l Schoenflies Conj is open

---

Pf of Alex Thm. We instead work in $M = \mathbb{R}^3$ and we assume $S \subseteq \mathbb{R}^3$ is a smooth two-sphere [cf Hatcher's Notes]

Perturb S so that the third coordinate, restricted to S is Morse. That is define

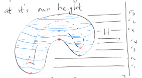
$$h: \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad (x,y,z) \longmapsto z$$

and define

$$H: S \longrightarrow \mathbb{R}, \quad p \longmapsto h(p)$$

i.e. $H = h|S$

We may assume that the critical pts of H are nondegenerate and each at it's own height

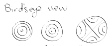


Def $\mathrm{Crit}(H) = \{\text{crit pts of } H\}$

Def $\mathrm{Reg}(H) = \{r_i\}_{i=1}^{n}$ where we choose r_i to lie between the heights of the i^{th} and $(i+1)^{th}$ critical points

Because H is Morse, near a crit pt we see one of

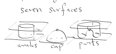
Bird's eye view

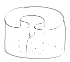


[Q Why can we perturb S^2?
A Morse thy, Isotopy extension]

Consider C, a component of $H^{-1}([r_i, r_{i+1}])$.

Claim: C is isotopic [level preserving] to one of the following seven surfaces





inside out [plus turning over the last three.]

[these three have more ∂-components below than they do above]

Pf Exercise in smooth topology. //

Furthermore they bound (in $h^{-1}([r_i, r_{i+1}])$) the correct thing.

Def The width of S is

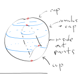
$$w(S) = \sum_{i=1}^{n} |H^{-1}(r_i)|$$

Ex

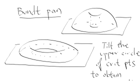
$w = 1$

$w = 4$

$w = 4$



Bundt pan



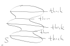
Tilt the upper circle of crit pts to obtain Morse position

Part the second

Fix $i \in \{1, 2, \dots, n\}$ a local max of $(j \longmapsto |H^{-1}(r_j)|)$

Call r_i a thick level

Cartoon



Pick $\alpha \subseteq H^{-1}(r_i)$ an innermost curve

That is, α bounds a disk D in $h^{-1}(r_i)$ st $D \cap S = \alpha$

We surger S along D to obtain two spheres S', S''

Perturb S', S'' along D', D'' only to add a cup in D', cup in D''

Case 1 $w(S') = 1$

ie S' look like the first example

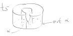
So [Ex] S' bounds a 3-ball B'

[Idea Cut into many slabs, straighten, apply $(J-S) \times I$, unstraighten using IET. At bottom and top use Morse thy

So. There is an isotopy of $S' - D'$, across B', to D'

Cartoon

Let A be the component of $H^{-1}([r_i, r_{i+1}])$ containing α as its lower boundary. Then A is one of annulus  or a cap  or a pants  or a inside out pants



Case 1a cap. So $S =$  is standard ✓

Generic picture

Case 1b. A is an annulus

Picture

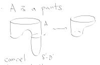
[Claim $\partial^+ A$ is also innermost]

Isotope $(S'-D') \cup A$ upward to just below height r_{i+1}

this reduces $w(S)$ [check!] [by at least two]

Case 1c A is a pants

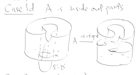
Picture



Here we cancel the saddle with the cup

This reduces $w(S)$

Case 1d A is inside out pants



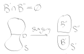
Case 2 $w(S'), w(S'') > 1$

Thus $w(S'), w(S'') < w(S)$

By induction, S', S'' bound 3-balls B', B''

Case 2a $B' \cap B'' = \emptyset$

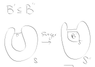
Cartoon,



Isotope $S'-D'$ across B' to get S isotopic to S''

Case 2b. $B' \subseteq B''$

Cartoon,



[Exercise If $B' \cap B'' \neq \emptyset$ then either $B' \subseteq B''$ or $B'' \subseteq B'$]

This completes the proof of Alexanders Thm $\square$

In fact this proves more

Thm If S, S' are embedded two-spheres in S^3 then S is isotopic to S'

To Ponder What is the homotopy type of $\mathrm{Emb}(S^2, S^3)$? [Smale conj (Hatcher)]

Ex Suppose $T \hookrightarrow S^3$ is a two-torus. Prove that T bounds a solid torus on at least one side

---

Thm [Kneser-Milnor]

Suppose M is comp. conn. oriented. Then M has a unique factorisation into primes

$$M = \#_i P_i$$