Lecture 4

Exercise Find an example or disprove:

(*) There is a compact, connected 3-mfd M st $\partial M \cong P^2$

Business

(1) Minsky Lectures webpage is part of EPSRC-Warwick Symposium page

(2) If you are taking the TCC for assessment please propose paper titles on Google group by next Wednesday

---

VII $\frac{1}{2}$ lives, $\frac{1}{2}$ dies

Recall Poincaré-Lefschetz duality

Fix M an n-manifold which is compact, connected and oriented. Then, working over a field K

$$H_i(M) \cong H_{n-i}(M, \partial M)$$

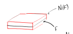
[Isomorphic as vector spaces over K. Also, the orientation is a K-orientation]

Q: Regarding the exercise: Surely ∂M is 2-sided so must be orientable

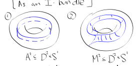
A: Consider the solid Klein bottle $\partial(SK) \cong K^2$. A2: Consider $P^2 \times I$

Def Suppose $(F^2, \partial F) \subset (M^3, \partial M)$ is properly embedded [and smooth and connected]. Let $N(F)$ be a tubular neighbourhood of F.

Picture



Thus $N(F) \to F$ is an I-bundle. Call F 2-sided if $\partial N(F)$ has two components, ie $N(F) \cong I \times F$ [As an I-bundle]



$A^2 \subseteq D^2 \times S^1$ two-sided

$M^2 \subseteq D^2 \times S^1$ one-sided

Recall If F is closed, oriented then $H_1(F) \cong K^{2g(F)}$, $g(F) = $ genus

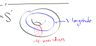
Lemma [$\frac{1}{2}$ lives, $\frac{1}{2}$ dies] Suppose M^3 is connected, compact, oriented. Set $F = \partial M$ [So F closed, oriented but not necess connected]

Let $i: F \hookrightarrow M$ be the inclusion. Then

$$\begin{aligned} \tfrac{1}{2}\dim(H_1(F)) &= \dim \ker(i_*) \quad \text{[dies]} \\ &= \dim \operatorname{Im}(i_*) \quad \text{[lives]} \end{aligned}$$

where $i_*: H_1(F) \to H_1(M)$

Ex: $M = \mathbb{D}^2 \times S^1$



Pf [of $\frac{1}{2}, \frac{1}{2}$]: Consider the LES of the pair (M, F) over K

If $F = \emptyset$ then we are done. Else, $H_3(F) \cong H_3(M) \cong 0$

$$0 \to H_3(M,F) \to H_2(F) \to H_2(M) \to H_2(M,F) \to H_1(F) \xrightarrow[i_*]{} H_1(M) \to H_1(M,F) \to H_0(F) \to H_0(M) \to 0$$

Recall:

$H_0(M) \cong H_3(M,F) \cong K$

$H_0(F) \cong H_2(F) \cong K^{|F|}$ # of components $|F| = p$

$H_1(M,F) \cong H_2(M) \cong K^n$

$H_1(M) \cong H_2(M,F) \cong K^m$

middle dim → $H_1(F) \cong H_1(F) \cong K^{2\Sigma g(F_i)} \cong K^{2\ell}$

Rewrite the LES

$$0 \to K \to K^p \to K^n \to K^m \to K^{2\ell} \to K^m \to K^n \to K^p \xrightarrow{\alpha} K \to 0$$

since α surjection, we can split the previous term as $K^{p-1} \oplus K$ where the first factor is the kernel. This works from both sides so strip off factors of the LES of the form $0 \to K \to K \to 0$ and find, in the centre,

$$0 \to K^\ell \to K^{2\ell} \xrightarrow[i_*]{} K^\ell \to 0$$ //

Q: What about $\mathbb{Z}$ coeff's?

A: Consider P^3

$H_3(P^3) \cong \mathbb{Z}$

$H_2(P^3) \cong 0$

$H_1(P^3) \cong \mathbb{Z}/2\mathbb{Z}$

$H_0(P^3) \cong \mathbb{Z}$

So do not get isom $H_1 \cong H_2$ as $\mathbb{Z}$-modules

Part Two

Corollary [DoubleCover]

Suppose M^3 is connected, compact

Suppose M has no conn double cover

Then (i) M is orientable

(ii) ∂M is a (perhaps empty) union of two-spheres

Pf If M is non-orientable then M has $M' \xrightarrow{2:1} M$ the orientation double cover. So suppose M is orientable. If some component $F \subset \partial M$ has genus then the lemma [½ ½] implies that $\dim_{\mathbb{Q}}(H_1(M)) > 0$. So $H_1(M,\mathbb{Z})$ surjects $\mathbb{Z}$ and thus surjects $\mathbb{Z}/2\mathbb{Z}$. So $\pi_1(M)$ surjects $\mathbb{Z}/2\mathbb{Z}$ and we are done //

VIII Knot complements

A knot $K \subset M$ is a locally flat [or PL, or smooth] (proper emb) embedding of S^1 into M.

This disallows wild knots

∞-many shrinking trefoils

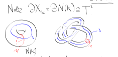
Define $N(K)$ = tubular neigh of K, thus $N(K)$ is a disk bundle over K

If K preserves orient then $N(K) \cong D^2 \times S^1$

" " reverses " " $N(K) \cong$ solid Klein bottle

Suppose $K \subset S^3$. Define $n(K) = \text{int}\, N(K)$ and $X_K = S^3 - n(K)$ = the knot exterior

Note $\partial X_K = \partial N(K) \cong T^2$

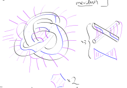


μ N(K) λ μ

By ½ lives ½ dies there is a unique curve $\lambda \subset T = \partial X_K$ which dies in $H_1(X_K)$

This is called the longitude

[μ dies in $H_1(N(K))$ – this is the meridian]



×3 ×2

Exercise The purple surface is homeo to $S_{1,1} \cong$

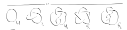
So shows that λ dies in $H_1(X_K)$

Def If λ is the longitude for K [$\lambda \subset \partial X_K$] then any orientable surf $(F, \partial F) \subseteq (X_K, \lambda)$ is called a Seifert surface. The minimal genus of such is called the genus of $K \subseteq S^3$, and denoted $g(K)$.

Rmk: We must prove that such surfaces exist [Eg via Seifert's Algorithm, or via more general techniques.]

Ex Compute the genus of the first few knots in Rolfsen's tables

---



U 3_1 4_1 5_1 5_2

6_1 etc [Story about Gabai and sutured mfds]

Open Is there a polynomial-time algorithm that, given $K \subset S^3$, computes $g(K)$? [Haken etc, This is exp time
HLP this is in NP
Lackenby this is in coNP]

Q How can we get lower bounds?

Eg perhaps $g(\text{trefoil}) = 0$?

Above we showed only $g(\text{trefoil}) \le 1$

[Ex. $g(K)=0$ iff $K \cong U$ (unknot)]

Examples:

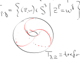
0) U the unknot

1) Torus knots

Fix $1 \le q \le p$ st $\gcd(p,q)=1$

Recall $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 2\}$

$T_{p,q} = \{(z,w) \in S^3 \mid z^p = w^q\}$



$T_{3,2}$ = trefoil

---

(3) Satellite knots

Fix $K \subset S^3$ a companion knot [not U]

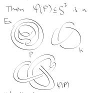
Fix $P \subset D^2 \times S^1$ a pattern knot

[P meets all disks isotopic to the meridian disk]

Fix $\varphi: D^2 \times S^1 \longrightarrow N(K)$ a homeo

Then $\varphi(P) \subseteq S^3$ is a satellite knot.

Ex



P K $\varphi(P)$

(4) all others

Thm [Thurston] Every knot in S^3 is either (1) U
(2) Torus knot
(3) Satellite knot or
(4) $\text{int}(X_K)$ has hyperbolic structure.
[complete, fin vol]