THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: APRIL 2019

GALOIS THEORY

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4, and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

COMPULSORY QUESTION

1. Throughout this problem, we will assume that L/K is a field extension.

a) (i) Without using the word "ring", define what it means for $\sigma\colon L \to L$ to be a *field automorphism* of L. **[3]**

(ii) Define what it means for σ to be a *relative automorphism* (also called a *K–automorphism*) of the extension L/K. **[2]**

(iii) Let $\mathrm{Aut}(L/K)$ be the set of relative automorphisms of L/K. Prove that $\mathrm{Aut}(L/K)$ is a group with respect to function composition. **[5]**

b) Suppose that $\sigma \in \mathrm{Aut}(L)$. Prove the following directly from the axioms for fields and the definitions given in part (a).

(i) $\sigma(0) = 0$, **[1]**

(ii) $\sigma(1) = 1$, **[1]**

(iii) $\sigma(-\alpha) = -\sigma(\alpha)$, for all $\alpha \in L$, and **[2]**

(iv) $\sigma(1/\alpha) = 1/\sigma(\alpha)$, for all non-zero $\alpha \in L$. **[2]**

c) Suppose that $H < \mathrm{Aut}(L/K)$ is a subgroup.

(i) Define what it means for $F = L^H = \mathrm{Fix}(H)$ to be the *fixed field* for H. **[1]**

(ii) Prove that F contains K. **[2]**

(iii) Prove that F is in fact a subfield of L. **[4]**

d) Suppose that $K \subset F \subset L$ is a tower of fields. Prove that $\mathrm{Aut}(L/F)$ is a subset of, and thus a subgroup of, $\mathrm{Aut}(L/K)$. (You may freely use the fact that both are subgroups of $\mathrm{Aut}(L)$.) **[4]**

e) (i) Define the *degree* of the extension L/K. **[1]**

(ii) Define what it means for L/K to be an *algebraic extension.* **[2]**

(iii) Prove that an extension with finite degree is algebraic. **[3]**

f) Fix $\alpha, \beta \in L$. Suppose that $K(\alpha)$ and $K(\beta)$ are algebraic extensions of K. Show that $K(\alpha, \beta)$ is also an algebraic extension of K. **[3]**

g) Suppose that $f \in K[x]$ is a polynomial.

(i) Define what it means for f to *split* in L. **[1]**

(ii) Define what it means for L to be a *splitting field* for f. **[2]**

h) Define what it means for L/K to be a *Galois extension.* **[1]**

OPTIONAL QUESTIONS

2. Set $f(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$.

a) Show that f is irreducible over $\mathbb{Q}$. **[4]**

Let α be the largest real root of f. For the remainder of this question we take $L = \mathbb{Q}(\alpha)$.

b) Show that $\{1, \alpha, \alpha^2\}$ is a basis for L, when thought of as a vector space over $\mathbb{Q}$. **[4]**

Suppose that β lies in L. Define $T_\beta \colon L \to L$ by $T_\beta(\gamma) = \beta\gamma$.

c) Show that T_β is a linear transformation of L, when thought of as a vector space over $\mathbb{Q}$. **[4]**

d) Use the basis $\{1, \alpha, \alpha^2\}$ to express T_α as a matrix. **[4]**

e) Let g be the characteristic polynomial of T_β.

(i) Prove that g lies in $\mathbb{Q}[x]$. **[2]**

(ii) Prove that β is a root of g. **[2]**

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3. Label each of the following claims TRUE or FALSE. For each, give a brief justification. No marks will be given for unjustified answers.

a) The quotient ring $R = \mathbb{C}[x]/(x^2 + 1)$ is a field. **[4]**

b) An extension is algebraic if and only if it has finite degree. **[4]**

c) Suppose that the extension L/K has degree two. Then it is normal. **[4]**

d) Suppose that the extension L/K has degree two. Then it is separable. **[4]**

e) The group $\mathrm{Aut}(\mathbb{R})$ is uncountable. **[4]**

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4. a) Suppose that $D \in \mathbb{Z}$ is positive and square-free (that is, if p is prime then p^2 does not divide D). Show that if $\sqrt{5}$ lies in $\mathbb{Q}\big(\sqrt{D}\big)$ then $D = 5$. **[4]**

For the remainder of this question we take $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Also, we take $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. You may freely appeal to any results from the lectures or the notes concerning $K/\mathbb{Q}$.

b) Show that $L/\mathbb{Q}$ is a Galois extension. **[4]**

c) Compute, with a careful justification, the degree and Galois group of $L/\mathbb{Q}$. **[6]**

d) Count the number of intermediate fields in $L/\mathbb{Q}$. Explain why your count is correct. **[6]**

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5. Set $\omega = \exp(2\pi i/7)$ and $L = \mathbb{Q}(\omega)$. Also, set $\alpha = \omega + 1/\omega = 2\cos(2\pi/7)$ and $K = \mathbb{Q}(\alpha)$.

a) Show that $L/\mathbb{Q}$ is a radical extension. **[3]**

b) Show that $L/\mathbb{Q}$ is a Galois extension. Find its degree and Galois group, with brief justifications. You may freely appeal to any results from the lectures or the notes. **[5]**

c) Show that $K/\mathbb{Q}$ is a Galois extension. Find its degree and Galois group, with brief justifications. You may freely appeal to any results from the lectures or the notes. **[6]**

d) Decide if $K/\mathbb{Q}$ is or is not a radical extension. Carefully justify your answer. **[6]**
