Please turn in Exercises 2.1, 2.2, and 2.6 before 2pm, on 26 October (Friday), in the slot near the front office. Please let me know if any of the problems are unclear, have typos, or have mistakes.

`http://homepages.warwick.ac.uk/~masgar/Teach/Current/class.html`

Exercise 2.1. Suppose that K is a field and that $f \in K[x]$ is a non-constant polynomial. Show that f is irreducible if and only if the ideal $(f) \subset K[x]$ is maximal.

Exercise 2.2. For any integer n we define $f_n(x) = x^3 + x + n \in \mathbb{Q}[x]$.

- Prove that f_n has exactly one root in $\mathbb{R}$.
- For each integer n, prove that f_n is irreducible over $\mathbb{Q}$ or give its factorisation. [A computer algebra system (CAS) such as SageMath may be useful in forming conjectures.]

Exercise 2.3. Suppose that $p \in \mathbb{N}$ is prime. Show that the polynomial

$$f_p(x) = x^{p-1} + x^{p-2} + \ldots + x + 1 \in \mathbb{Q}[x]$$

is irreducible over $\mathbb{Q}$.

Exercise 2.4. Suppose that $p \in \mathbb{N}$ is prime. Let $K = \mathbb{F}_p(t)$ be the field of rational functions with coefficients in the finite field $\mathbb{F}_p$. Show that $f(x) = x^n + t \in K[x]$ is irreducible over K.

Exercise 2.5. For $n = 1, 2, \ldots, 16$ factor the polynomial $u_n(x) = x^n - 1$ into irreducibles over $\mathbb{Q}$. [Again, a CAS will be useful to check your work.] The new factor in each case is called a *cyclotomic* polynomial.

Exercise 2.6. Consider the subfield $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subset \mathbb{C}$. Give a direct proof that all elements of K are of the form

$$a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$$

where a, b, c, d are rational.