Please turn in Exercises 3.4, 3.5, and 3.6 before 2pm, on 9 November (Friday), in the slot near the front office. Please let me know if any of the problems are unclear, have typos, or have mistakes.

`http://homepages.warwick.ac.uk/~masgar/Teach/Current/class.html`

Exercise 3.1. Recall that $\mathbb{F}_2$ is the finite field of two elements. List all irreducible polynomials of degree at most four inside of $\mathbb{F}_2[x]$. Explain why your list is correct.

Exercise 3.2. [Very hard.] Set $f(x) = x^3 + x + 1 \in \mathbb{Q}[x]$. Let α be the real root of f. For all $n \in \mathbb{Z}$ find a minimal polynomial for α^n. A computer algebra system *may* be useful for making conjectures.

Exercise 3.3. [Hard.] Show that $f_n(x) = x^n + x + 3$ is irreducible for all $n \in \mathbb{N}$.

Exercise 3.4. Prove that $\overline{\mathbb{Q}}$ is countable. Now show that $[\overline{\mathbb{Q}} : \mathbb{Q}]$ is countably infinite.

Exercise 3.5. Suppose that L is an extension of K with $[L : K] = p$ a prime. Prove that L is a simple extension of K.

Exercise 3.6. A field K is *quadratic* if it is a quadratic extension of $\mathbb{Q}$; that is, if K contains $\mathbb{Q}$ and $[K : \mathbb{Q}] = 2$. Prove that if K is quadratic then there is a square-free integer D so that $K \cong \mathbb{Q}(\sqrt{D})$.

For the next two problems we take $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Exercise 3.7. Give a careful proof that $[L : \mathbb{Q}] = 4$.

Exercise 3.8. [Medium.] Find all subfields of L and give a proof that your list is complete. [Hint: Use the tower law as well as Exercises 2.6, 3.6, and 3.7.]