Please turn in Exercises 5.2 and 5.3 before 2pm, on 7 December (Friday), in the slot near the front office. Please let me know if any of the problems are unclear, have typos, or have mistakes.

Exercise 5.1. Let $\sigma\colon \mathbb{C} \to \mathbb{C}$ be complex conjugation. Give an example of a field $L \subset \mathbb{C}$ so that $\sigma(L) \neq L$.

Exercise 5.2. Let $\sigma\colon \mathbb{C} \to \mathbb{C}$ be complex conjugation. Suppose that $L \subset \mathbb{C}$ is a subfield, and suppose that $L/\mathbb{Q}$ is a finite Galois extension.

1. Show that $\sigma(L) = L$.
2. We define $\sigma_L = \sigma|L$. Show that $\sigma_L \in \mathrm{Gal}(L/\mathbb{Q})$. Show that σ_L has order one or two.
3. Show that σ_L has order one if and only if $L \subset \mathbb{R}$.
4. Let $F = L^{\langle \sigma_L \rangle}$. Show that the degree $[L : F]$ is one or two exactly as L is or is not contained in $\mathbb{R}$.

Exercise 5.3. Consider the polynomial $f(x) = x^4 - 2 \in \mathbb{Q}[x]$.

1. Show that $L = \mathbb{Q}(i, \sqrt[4]{2})$ is a splitting field for f. Deduce that $L/\mathbb{Q}$ is Galois.
2. Compute $[L : \mathbb{Q}]$ and $\mathrm{Gal}(L/\mathbb{Q})$. Express all elements of the latter in terms of their action on the roots of f.
3. Let $K = \mathbb{Q}(\sqrt{2})$. Give a basis $\{\alpha_i\}$ for $K/\mathbb{Q}$. Give a basis $\{\beta_j\}$ for L/K.
4. Let τ be the non-identity element of $\mathrm{Gal}(K/\mathbb{Q})$. Show that, for any $\sigma \in \mathrm{Aut}(L/\mathbb{Q})$, if $\sigma|K = \tau$ then $\sigma|\{\beta_j\}$ is not the identity.

Exercise 5.4. Prove that $\alpha = 3 + 4i$ is a square in $\mathbb{Q}(i)$, but $\beta = 2 + i$ is not.

Exercise 5.5. Suppose that L/K is a finite Galois extension. For any $\alpha \in L$ we define its *norm* to be

$$\mathrm{Norm}_{L/K}(\alpha) = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(\alpha)$$

1. Show that $\mathrm{Norm}_{L/K}(\alpha)$ lies in K.
2. Show that $\mathrm{Norm}_{L/K}(\alpha\beta) = \mathrm{Norm}_{L/K}(\alpha) \cdot \mathrm{Norm}_{L/K}(\beta)$.
3. Suppose that $L = K(\alpha_1, \ldots, \alpha_n)$. Show that $L = K[\alpha_1, \ldots, \alpha_n]$. (That is, the ring generated by the α_i equals the field generated by the α_i; division is unnecessary!)