MA4J70

THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: MAY 2019

COHOMOLOGY AND POINCARÉ DUALITY

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4, and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

COMPULSORY QUESTION

1. In parts (a) through (g), assume that A and G are abelian groups.

a) Give the definition of $\mathrm{Hom}(A, G)$ and briefly show that it is an abelian group. **[3]**

b) Give the definition of a *free resolution* of A. **[3]**

c) Give an example of an abelian group A and a pair of non-isomorphic free resolutions of A. **[3]**

d) Give the definition of $\mathrm{Ext}(A, G)$. **[3]**

e) Sketch a proof that $\mathrm{Ext}(A, G)$ is well-defined. **[5]**

f) Prove the following.

(i) If A is free, then $\mathrm{Ext}(A, G) \cong 0$. **[3]**

(ii) If $A = \mathbb{Z}/n\mathbb{Z}$, then $\mathrm{Ext}(A, G) \cong G/nG$. **[3]**

(iii) $\mathrm{Ext}(A \oplus B, G) \cong \mathrm{Ext}(A, G) \oplus \mathrm{Ext}(B, G)$. **[3]**

g) State the universal coefficient theorem. Include brief descriptions of all homomorphisms that appear. **[5]**

In parts (h) through (j), assume that K is the Klein bottle and $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

h) Compute the homology groups $H_k(K)$, for all k, giving a brief justification of your answers. **[3]**

i) Use the universal coefficient theorem to compute the cohomology groups $H^k(K; \mathbb{Z})$, for all k, giving a careful justification of your answers. **[3]**

j) Use the universal coefficient theorem to compute the cohomology groups $H^k(K; \mathbb{F})$, for all k, giving a careful justification of your answers. **[3]**

OPTIONAL QUESTIONS

2. a) Give a CW–complex structure for the two-sphere S^2. Use it to give explicit generators for the cohomology ring $H^*(S^2; \mathbb{Z})$. **[4]**

b) Using the generators from part (a), describe the cup product structure on $H^*(S^2; \mathbb{Z})$ and briefly justify your answer. **[4]**

c) Give a CW–complex structure for the four-manifold $X = S^2 \times S^2$. Use it to give explicit generators for the cohomology ring $H^*(X; \mathbb{Z})$. **[6]**

d) Using the generators from part (c), describe the cup product structure on $H^*(X; \mathbb{Z})$ and carefully justify your answer. **[6]**

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3. a) Give the definitions of

(i) a *directed set*,

(ii) a *directed system* of abelian groups,

(iii) and the *direct limit* of a directed system of abelian groups. **[6]**

Let $S_{g,b}$ be the compact, connected, $\mathbb{Z}$–oriented surface (two-manifold) of genus g and with b boundary components.

b) Show, by means of a labelled figure, that gluing a copy of $S_{1,2}$ to a copy of $S_{g,1}$ along a boundary component of each results in a copy of $S_{g+1,1}$. **[4]**

c) Let $\iota\colon S_{g,1} \to S_{g+1,1}$ be the inclusion described in part (b). Show that the homomorphism $\iota_*\colon H_1(S_{g,1}) \to H_1(S_{g+1,1})$ is injective; show that its cokernel is a free abelian group. **[6]**

d) Let S_∞ be the ascending union $\cup_g S_{g,1}$. Compute $H_1(S_\infty)$ and briefly justify your answer. **[4]**

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4. Let $D = D^2$ be the closed unit disk in the plane. Let $V = S^1 \times D$ be the closed *solid torus*. Let ∂D to be the boundary of D. Let $\partial V = S^1 \times \partial D$ be the boundary of V.

a) Give a CW-complex structure for V. (Choosing a simple structure here will make the following problems easier.) **[6]**

b) Draw a picture of V. Clearly label the boundary ∂V and the CW-complex structure of V. **[2]**

c) Compute the relative cohomology groups $H^k(V, \partial V; \mathbb{Z})$ for all k. Carefully justify your answers. **[6]**

d) Let $V^\circ = V - \partial V$ be the *interior* of V. Compute the cohomology with compact supports $H^k_c(V^\circ; \mathbb{Z})$ for all k. Carefully justify your answers. **[6]**

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5. In this problem we use $\mathbb{Z}$ coefficients throughout and thus suppress them from the notation.

Suppose that M is a compact connected oriented n–manifold without boundary. Suppose that $U \subset M$ is an open chart in M; that is, U is an open subset equipped with a homeomorphism $\phi\colon U \to \mathbb{R}^n$. Let $D \subset M$ be the preimage under ϕ of the closed unit n–ball in $\mathbb{R}^n$. We call D a *standard ball* in M. Let D° be the interior of D; that is, the preimage under ϕ of the open unit n–ball.

Finally, let N be another compact connected oriented n–manifold without boundary. Let $E \subset N$ be a standard ball in N. You may assume that the $\mathbb{Z}$–algebras $H^*(M)$ and $H^*(N)$ are known.

a) Set $M_D = M - D^\circ$. Compute the cohomology groups $H^k(M_D)$, for all k. **[5]**

b) Define the *connected sum* $X = M \# N$. **[3]**

c) Prove that X is orientable. **[6]**

d) Compute the cohomology groups $H^k(X)$, for all k; briefly justify your answer. **[3]**

e) Describe the cup product structure on $H^*(X)$; briefly justify your answer. **[3]**
