Please let me know if any of the problems are unclear, have typos, or have any other mistakes.

Exercise 2.1. Find a cell-structure on $\mathbb{RP}^n$. Using this, or otherwise, compute the homology groups of $\mathbb{RP}^n$.

Exercise 2.2. Find a cell-structure on $\mathbb{CP}^n$. Using this, or otherwise, compute the homology groups of $\mathbb{CP}^n$.

Exercise 2.3. Suppose that $f\colon S^n \to S^n$ is a map. Prove that the degrees of the homomorphisms

$$f_n\colon H_n(S^n) \to H_n(S^n) \qquad \text{and} \qquad f^n\colon H^n(S^n;\mathbb{Z}) \to H^n(S^n;\mathbb{Z})$$

are equal.

Exercise 2.4. Compute the homology groups of the following spaces.

1. $X = (S^n)^m$. [More care is required when $n = 1$.]
2. $X = S^n \times S^m$.

Exercise 2.5. Suppose that X is a topological space. Suppose that $A, B \subset X$ are subsets so that X is contained in the union of the interiors of A and B. Suppose that G is an abelian group. Let $C_*(A+B)$ be the chain complex consisting of all singular chains subordinate to the "cover" $\{A, B\}$. In the proof of excision we showed that the inclusion

$$i\colon C_*(A+B) \to C_*(X)$$

has a chain homotopy inverse ρ. Using this, or otherwise, prove that the dual homomorphism

$$i^\#\colon C^*(X;G) \to C^*(A+B;G)$$

has a chain homotopy inverse.

Exercise 2.6. With notation as in the previous problem. The Mayer-Vietoris long exact sequence, in homology, comes from the short exact sequence of chain complexes

$$0 \to C_*(A\cap B) \xrightarrow{\Delta} C_*(A)\oplus C_*(B) \xrightarrow{m} C_*(A+B) \to 0$$

Here the homomorphisms are given by $\Delta(c) = (c, c)$ and $m(c, d) = c - d$. Prove that the dual sequence

$$0 \leftarrow C^*(A\cap B;G) \xleftarrow{\Delta^\#} C^*(A;G)\oplus C^*(B;G) \xleftarrow{m^\#} C^*(A+B;G) \leftarrow 0$$

is exact.