Intro to three-manifolds Saul Schleimer 1

1. overview: Start with many definitions and examples from Thurstons geometric thy of 3-mfd.

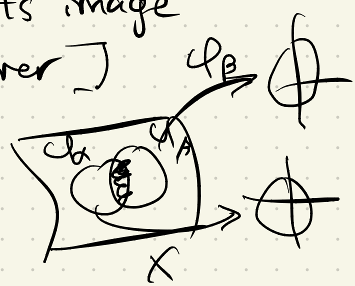
Then move on to cut and paste topology and prove the sphere, disk, torus, and annulus theorems. There are various proofs of these [Stallings, Perelman, ... we will follow notes of Casson]

2. Admin: Every thing is on class webpage see link in the chat.

Exercises: If you are taking course for credit please answer one question every two weeks.

I. Manifolds: Suppose X is a topological space. We say X is an n-manifold if it is Hausdorff, second countable, and has a topological atlas $\{(\varphi_\alpha, U_\alpha)\}_\alpha$: that is

- $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$
- φ_α is a homeomorphism to its image
- $X = \bigcup \{U_\alpha\}_\alpha$ [U_α cover]
- U_α are all open sets
- $\varphi_\alpha \circ \varphi_\beta^{-1} \big| \varphi_\beta(U_\alpha \cap U_\beta)$ is a homeomorphism.



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<u>Notation</u> $(\varphi_\alpha, U_\alpha)$ is called a <u>chart</u> and $\{(\varphi_\alpha, U_\alpha)\}$ is called an <u>atlas</u>.

We call $\varphi_\alpha \circ \varphi_\beta^{-1} | \varphi_\beta(U_\alpha \cap U_\beta) = \Phi_{\alpha\beta}$ an <u>overlap map</u> [or a <u>transition map</u>]

<u>Example</u>: $\mathbb{R}^n$ is an n-manifold.

<u>Definition</u>: The <u>n-sphere</u> is the subspace
$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}.$$

<u>Exercise</u>: Prove S^n is an n-manifold.

<u>Definition</u>: $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ [also call this $\mathbb{D}^n$]

This is the <u>n-ball</u> [also called the n-disk]

<u>Notation</u>: $I = [0,1] \cong \mathbb{B}^1 = \mathbb{D}^1$ [$\cong$ homeomorphic]

<u>Exercise</u>: $\mathbb{B}^n$ is <u>not</u> an n-manifold.

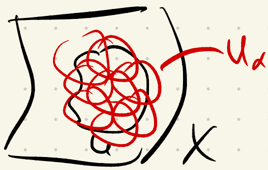
II (G, X)-structures after Thurston.

Fix X a top. space. Fix G a <u>pseudo group</u> on X. That is $G = \{\varphi_\alpha : U_\alpha \to V_\alpha\}_\alpha$ is a set of homeomorphisms so that

- $U_\alpha, V_\alpha \subset X$ are open
- $\bigcup \{U_\alpha\}_\alpha = X$
- If $U \subset U_\alpha$ then $\varphi_\alpha | U \in G$
- If $\varphi_\alpha \circ \varphi_\beta$ is defined then it lies in G [that is if $V_\beta \subset U_\alpha$]
- $\varphi_\alpha^{-1} \in G$
- Suppose $\varphi : U \to V$ is a homeo, with $U, V \subset X$ open. If there are (φ_i, U_i) with U_i covering U and φ_i's agreeing

on intersections, Then $\varphi \in G$

Picture



[**Definition**: Fix Y top space. A (G,X) str on Y is a G-atlas $\{(\varphi_\alpha, U_\alpha)\}$: The U_α cover Y and the transition fcns lie in G. See page 110 of Thurston.]

Example: Suppose $\varphi : U \to V$ is a homeo of open subsets of $\mathbb{R}^n$. We define

$$\varphi_x : H_{n-1}(U, U-x ; \mathbb{Z}) \cong \mathbb{Z}$$
$$\downarrow$$
$$H_{n-1}(V, V-\varphi(x) ; \mathbb{Z}) \cong \mathbb{Z}$$

This is an isomorphism. There is a "canonical" generator of both of these groups $\left[\begin{array}{l} H_{n-1}(U, U-x) \cong \\ H_{n-1}(\mathbb{R}^n, \mathbb{R}^n - x) \end{array} \right]$

We call φ orientation preserving if φ_x preserves the canonical gen (for all x).

Define $\underset{\text{orient pres.}}{\mathrm{Top}^+(\mathbb{R}^n)} < \underset{\text{pseudo group of homeomorphisms.}}{\mathrm{Top}(\mathbb{R}^n)}$

Definition: M is an oriented n-manifold if it has an atlas of charts $\{(\varphi_\alpha, U_\alpha)\}$ with transitions in $\mathrm{Top}^+(\mathbb{R}^n)$.

Exercise: S^n is an orientable manifold.

Definition: Set $X = \mathbb{R}^n_{\geq 0} = \{ x \in \mathbb{R} : x_n \geq 0 \}$. Set $G = \mathrm{Top}(X)$. A (G,X) space is an n-manifold with boundary.

Exercise: B^n is a manifold with boundary.

Exercise: Define $\mathrm{Top}^+(\mathbb{R}^n_{\geq 0})$ and thus oriented mfds w/ ∂.

Definition: Suppose M^n is mfd w/∂. Define

$\partial M = \{x \in M \mid x$ does not have a chart to $\mathbb{R}^n\}$

Exercises: (1) ∂M is an $n-1$ mfd

(2) $\partial(\partial M) = \phi$

Definition: A manifold M is closed if M is compact and $\partial M = \phi$.

Exercises: (1) S^n is closed

(2) $\partial \mathbb{B}^n = S^{n-1}$

(3) $\mathbb{R}^n, \mathbb{B}^n$ are not closed

(4) S^n has two incompatible orientations.

$\nparallel$

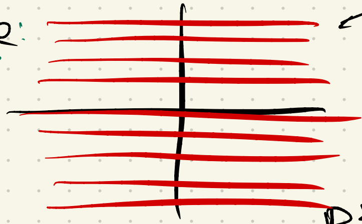
Question: what is the canonical generator for $H_{n-1}(U, U-x; \mathbb{Z})$ for $U \subset \mathbb{R}^n$ open?

Answer: $H_{n-1}(\mathbb{R}^n, \mathbb{R}^n - x) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$

This gives me a generator for $H_{n-1}(\mathbb{R}^n, \mathbb{R}^n - x)$ for all x. Now the excision isomorphism gives a generator of $H_{n-1}(U, U-x)$.

Question: What is a k-dim'l foliation of an n-dim'l manifold?

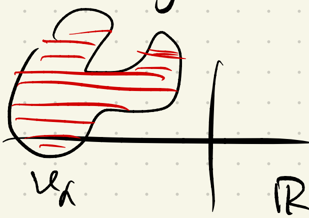
Answer: Define the standard foliation (of dim k) of $\mathbb{R}^n$ to be the partition $\{\mathbb{R}^k \times (0, \dots 0, x) \mid x \in \mathbb{R}^{n-k}\}$

Picture:  $\mathbb{R}^2$

← The parts are called leaves of the foliation. This is 1-dim foliation in $\mathbb{R}^2$

Picture  } 2-dim'l foliation on $\mathbb{R}^3$.

Define $G_{k,n} < \mathrm{Top}(\mathbb{R}^n)$ to be the pseudo group preserving the standard foliation.



u_α $\mathbb{R}^2$.

$\varphi_\alpha : U_\alpha \to V_\alpha$ must send leaves to leaves.

Definition: A k-dim foliation on a n-manifold M is a $G_{k,n}$ structure.

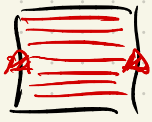
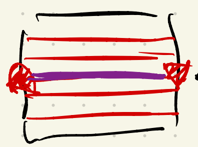
Example: Suppose $M^n = N^k \times P^{n-k}$ is a product of manifolds. Then M^n has a k-foliation where all leaves are homeo to N. This is a product foliation.

Picture: $A^2 = S^1 \times I$, $M = S^1 \tilde{\times} I$

Exercise: M^2 is not orientable

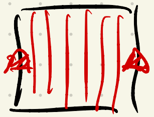
all leaves are circles

product foliation

central leaf is half as long as others!

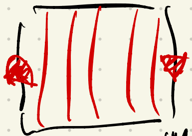
If we generalise to manifolds with boundary we can define foliations with boundary.



A^2

orientable foliation

$(G^+_{k,n})$



M^2

nonorientable foliation.

This gives a foliation by intervals instead of circles.

Exercise: Give an example of a nonorientable foliation in an orientable manifold.

Question: We saw →
what about in manifolds with corners?

mfds ↑ \ foliations →	$\partial=\emptyset$	$\partial\neq\emptyset$
$\partial=\emptyset$	✓	DNE
$\partial\neq\emptyset$	✓	✓

III Classification of manifolds

Problem: classify n-manifolds up to homeomorphism

$\boxed{n=0}$ These are classified by there cardinality

B^0 $\quad$ S^0 $\quad$ ...

Now restrict to connected manifolds.

$\boxed{n=1}$

S^1 $\quad$ $I\cong B^1$ $\quad$ $\mathbb{R}^1$ $\quad$ $\mathbb{R}_{\geq 0}$

circle $\quad$ interval $\quad$ line $\quad$ ray

Now restrict to compact manifolds

$\boxed{n=2}$ **Definition** $\left.\begin{array}{l}\mathbb{T}^0 = \{pt\} = \mathbb{B}^0 \\ \mathbb{T}^{n+1} = S^1 \times \mathbb{T}^n\end{array}\right\}$ n-torus.

Pictures

$\mathbb{T}^0$ $\mathbb{T}^1$ $\mathbb{T}^2$ $\mathbb{T}^3$ glue opposite sides by translation . . .

Definition: $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{R}} = \mathbb{RP}^n = \mathbb{P}(\mathbb{R}^{n+1}) = \mathbb{P}_n(\mathbb{R})$

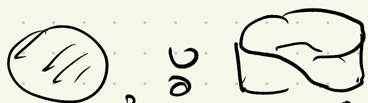
Fix K a field. Define $\mathbb{P}^n_K = \dfrac{K^{n+1} - \{0\}}{K - \{0\}}$

$=$ space of lines in K^{n+1}

i.e. $\mathbb{P}^n_K$ is the quotient of $K^{n+1} - \{0\}$ where $x \sim y$ iff $y = \lambda x$, $\lambda \in K - \{0\}$.

Exercise: $\mathbb{CP}^1 \cong S^2$.

Exercise: Draw $\mathbb{RP}^2$:



$D^2 \cup M^2$

Next time: Connect sums

Classification of surfaces.

Question: Given X, is it possible to classify foliations on X?

Answer: Foliations can be pretty wild! So you need extra hypotheses to have a sensible theory.

Exercises: (1) There are no 1-dim'l foliations on S^2

(2) "Classify" foliations on $\mathbb{T}^2$

(3) Find a 2-dim'l foliation on S^3. [or 1-dim'l]