Three-manifolds Lecture II 2021-01-27

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Last time: manifolds, (G,X) structures
classification of n-mfds ($n=0,1$)

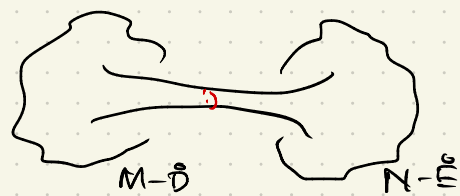
(I) Connect sums and sphere surgery

Suppose M, N are n-mfds (conn, oriented). Fix $D \subset M$, $E \subset N$ n-balls (round in some chart) Picture

Fix $\varphi : \partial D \to \partial E$ orient reversing homeomorphism.

We define the connect sum $M \# N = (M - \mathring{D}) \sqcup (N - \mathring{E}) / \varphi$

Picture



Surgery is the opposite: cut along S^2 and cap off with B^3's.

Lemma Suppose S is conn, orient, surface. Then $S \# S^2 \cong S$.

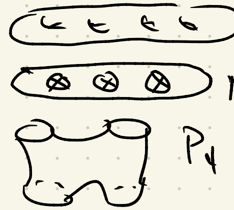
Proof: Jordan-Schönflies Theorem.
Now use Alexander trick
$\mathrm{Homeo}(D^2, \partial D) \simeq \{\mathrm{Id}\}$. //

Exercises: $M \# N \cong N \# M$, $(M \# N) \# P \cong M \# (N \# P)$

Definition $S_g = gT^2 = \#_g T^2$

$N_c = c\mathbb{P}^2 = \#_c \mathbb{P}^2$

$P_b = bD^2 = \#_b D^2$



Terminology: S_g is the surface of genus g

N_c " " " " non-ori " c

P_b is the planar surface with b boundary components.

Exercise: (1) $\chi(F \# G) = \chi(F) + \chi(G) - 2$ [F, G surfaces]

(2) $\chi(S_g) = 2 - 2g$

(3) $N_3 \cong T^2 \# \mathbb{P}^2$.

(4) $S_g \xrightarrow{\times 2} N_{g+1}$ is ori double cover.

II Classification:

$\boxed{n=2}$ **Theorem** [Radó ...] Every cpt. conn. surf. is homeo. to some $S_g \# N_c \# P_b$.

Corollary: A cpt. conn surface S is det. by

(1) orientability

(2) $\chi(S)$

(3) $|\partial S|$ = # of bound. components.

Proof Exercise. //

$\boxed{n=4}$ Undecidable! There is no classification of conn, cpt four-manifolds.

$\boxed{n=3}$ **Geometrisation** [Thurston, Hamilton, Perelman, ...] Suppose M is a closed, conn, irreducible, atoroidal three-manifold. Then M admits one of the eight (well, four) Thurston geometries.

Corollary The homeo. problem for cpt, conn 3-mfds is decidable.

III overview of Thurston geometries

A table

Seifert fibered geometries	S^3	Nil	$\widetilde{PSL}$	twisted S^1 bundles over surfaces
	$S^2 \times \mathbb{R}$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{R}$	product geom
		Sol	$\mathbb{H}^3$	"twisted surf. bundles over S^1"
	"spherical"	Torus bundles over S^1	"hyperbolic"	

IV S^3-geometry [elliptic mfds aka spherical space forms]

Let $X = S^3$ with the round metric.

Let G be the pseudo group gen. by $\mathrm{Isom}^{\pm}(S^3)$.

Call a (G,X)-structure on M complete if the induced metric on M is complete.

Bad example: $S^3 - \{pt\}$.

Standing assumption: From now on (G,X) structures (for geometries) will be complete.

Exercise: Elliptic manifolds are orientable.

In particular: $\mathbb{RP}^3$ is orientable.

Theorem: If M is elliptic (closed, complete) Then there is some $\Gamma < SO(4) \cong \mathrm{Isom}^+(S^3)$ so that $M \cong S^3/\Gamma$. [Conclude Γ is finite, acts nicely on S^3.]

[Ref: Scott's Bull. AMS article]

Examples: Let $R(\theta) \in SO(2)$ be the rotation thru angle θ. Fix $1 \le q \le p$ with $\gcd(p,q) = 1$.

Define $\varphi = \varphi_{pq} = \begin{pmatrix} R(2\pi/p) & 0 \\ 0 & R(2\pi q/p) \end{pmatrix} \in SO(4)$

Define $L(p,q) = S^3/\langle\varphi\rangle$ the (p,q)-<u>lens space</u>

Exercises: (1) $L(1,1) \cong S^3$

(2) $L(2,1) \cong \mathbb{RP}^3 \cong SO(3) \cong \mathrm{Isom}^+(S^2) \cong UTS^2$

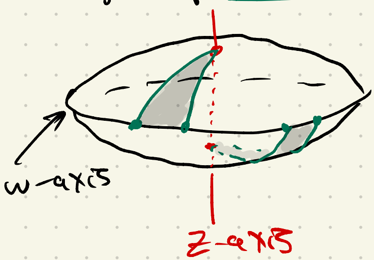
[UT = <u>unit tangent bundle</u>]

(3) $\pi_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$

(4) Classify lens spaces up to homeo (up to h.e.)

Origin of the name: Draw $\mathbb{B}^3$ as a lens

Glue the top to the bottom by a $2\pi q/p$ rotation, $\phi_{p,q}$



[$S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$ spanned by z and w planes]

[z and w axes are unit circles in those planes.]

Exercise

$\mathbb{B}^3/\phi \cong L(p,q)$. <u>Hint</u>: Take x on z axis and $\mathcal{O}_x = \langle\varphi\rangle \cdot x$ orbit of x

Show Voronoi domains are lenses

Challenge: What if x is not on the z or w-axes in S^3? [Then dualize to get triangulation.]

Exercise: Every elliptic mfd has a (at most 60-fold) cover by a lens space.

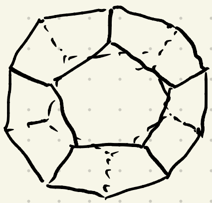
Definition: Suppose $\Gamma < SO(3)$. Define
$\Gamma^* < S^3 \cong SU(2) \cong$ unit quaternions
to be the binary extension of Γ:

$$\begin{tikzcd} \Gamma^* \arrow[d] & < & SU(2) \arrow[d, "\times 2"] \\ \Gamma & < & SO(3) \end{tikzcd}$$

If Γ = sym of dodecahedron
Then Γ^* is the binary version.

Define: $PHS^3 = S^3/\Gamma^*$

Picture:



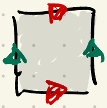
[Glue opposite faces by a right handed $1/10$, $3/10$, or $5/10$ twist
This gives three diff mfds. What are they?]

[Variant of this gives other SFS's with base orbifold $S^2(2,3,5)$]

V Surface bundles: Suppose F is cpt conn surface.

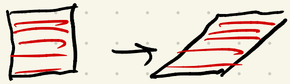
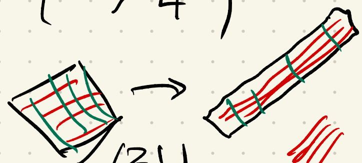
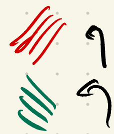
Def: A homeo $f: F \to F$ is

- periodic (elliptic) if there is some k so that $f^k = Id_F$ [or just isotopic to Id_F]
- reducible (parabolic or pseudo-hyperbolic) if there is a multi curve $C \subset F$ with $f(C) = C$ [or just isotopic]
- (pseudo) Anosov [hyperbolic] if there are transverse (singular) foliations $\mathcal{F}^{\pm}$ in F and $\lambda > 1$ st. $f(\mathcal{F}^{\pm}) = \mathcal{F}^{\pm}$ and stretches or shrinks them by a factor of λ.

Examples: $F = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  If $A \in SL(2,\mathbb{Z})$, $A : \mathbb{R}^2 \circlearrowleft$ then define $f = f_A$

to be the resulting homeo on the quotient

f_A is $\left\{ \begin{array}{l} \text{periodic} \\ \text{reducible} \\ \text{Anosov} \end{array} \right\}$ iff $(\operatorname{tr} A)^2$ is $\left\{ \begin{array}{l} < 4 \\ = 4 \\ > 4 \end{array} \right\}$

Eg:  1/4   

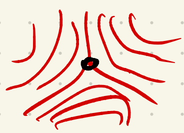
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ rotate

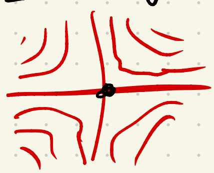
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ shear

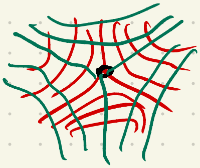
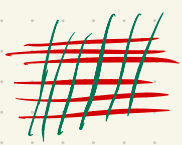
$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ hyperbolic.

eigen foliations

Def: $\mathcal{F}$ is a singular foliation if it is a foliation away from a finite st $Z \subset F$ and at $z \in Z$ we see a singularity

 3-pronged singularity.

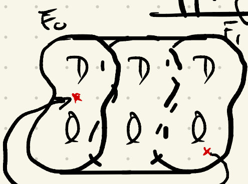
 4-pronged singularity

  Transverse foliations.

Exercise: Give explicit example of a pseudo Anosov map

Definition: Suppose F is a surface, $f : F \to F$ a homeomorphism.

The mapping torus $M_f = F \times [0,1] / (x,1) \sim (f(x),0)$



This is equipped with two foliations. the foliation $\{F \times \{t\}\}_{t \in I}$ (2-dim'l) and the foliation $\{\{x\} \times I\}_{x \in F}$ (1 dim'l).

Terminology: Also call M_f a surf. bundle over S^1 with fiber F and monodromy f.

Examples: If $f = \mathrm{Id}_F$ then $M_f = F \times S^1$

Example: If $f : S^2 \to S^2$ is a reflection then $M_f = S^2 \hat{\times} S^1$ is the "twisted S^2 bundle over S^1".

Question: Can we express triangulations (or other combinatorial decompositions) in terms of (G, X) structures?

Answer: I think I see how to express "cubic graphs on surfaces" as (G, X)-structures. Consider the following pictures:

But we cannot control the complementary regions this way...