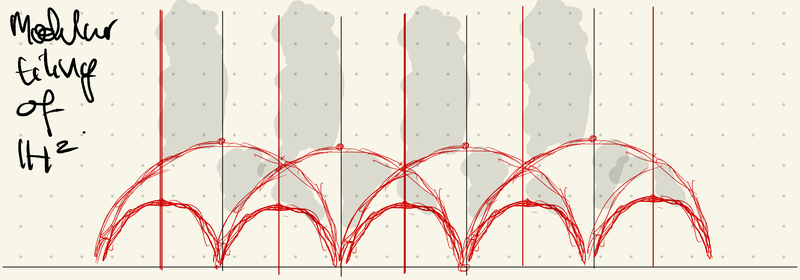
Modular tiling of $\mathbb{H}^2$



Intro to 3-mflds

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Lecture 3

Saul Schleimer

Last time:
- Connect sums
- Classification of manifolds
- overview of Thurston geometries
- S^3 geometry

(I) $S^2 \times \mathbb{R}$ geometry: $X = S^2 \times \mathbb{R}$ (usual metric)

Take $G \doteq \mathrm{Isom}(X) \cong \mathrm{Isom}(S^2) \times \mathrm{Isom}(\mathbb{R})$

The manifolds with (G, X) structures are exactly

noncompact: $S^2 \times \mathbb{R}$, $P^2 \times \mathbb{R}$, $P^3 - \{pt\}$

compact: $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, $P^2 \times S^1$, $P^3 \# P^3$ $[P^3 = \mathbb{RP}^3]$

twisted S^2 bund / S^1. connect sum

Exercise: Find all covering maps among theses.

(II) Torus bundles: Set $F = \mathbb{T}^2$, $A \in SL(2, \mathbb{Z})$

Define $M_A = \mathbb{T}^2 \times I / (x,1) \sim (Ax, 0)$. Then

$$M_A \text{ has geometry } \left\{ \begin{matrix} \mathbb{E}^3 \\ \mathrm{Nil} \\ \mathrm{Solv} \end{matrix} \right\} \text{ iff } A \text{ is } \left\{ \begin{matrix} \text{periodic} \\ \text{reducible} \neq \mathrm{Id} \\ \text{Anosov} \end{matrix} \right\}$$

Proposition: If M has $\mathbb{E}^3$, Nil, or Solv geom then M is (at most four fold) covered by such a torus bundle. [M compact]

Example: 

$2\pi/6$ rotation

There are five $\mathbb{E}^3$ torus bundles [fin. ord elts of $SL(2,\mathbb{Z})$]

Exercise: Find the last orientable $\mathbb{E}^3$ mfd.

Example: Set $H(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}$

Define $H(\mathbb{Z})$ similarly. Then $H(\mathbb{R})/_{H(\mathbb{Z})} \cong M_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}$

Rmk: $Solv \cong \mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ is 3-dim lie group. Nil.

$$t \cdot (x,y) = (e^t x, e^{-t} y)$$

III Hyperbolic surfaces:

$F = S_2^3$

Suppose $F = S_g^p$ is a hyperbolic surface with fin volume genus g and punctures p. Suppose $f: F \to F$ is a homeomorphism. Then:

M_f is $\left\{ \begin{array}{l} \mathbb{H}^2 \times \mathbb{R} \text{ geom} \\ \underline{\text{toroidal}} \\ \mathbb{H}^3 \text{ geom} \end{array} \right\}$ iff f is $\left\{ \begin{array}{l} \text{periodic} \\ \text{reducible} \\ \text{pseudo Anosov} \end{array} \right\}$

These are respectively exercise, exercise, Thurston's double limit theorem. [Thurston, Otal, Perelman ...]

Exercise: Any closed orn $\mathbb{H}^2 \times \mathbb{R}$ geom manifold is (at most four fold) covered by a bundle as above.

Theorem [Agol, Kahn Markovic, Manning, Wise ..]
Any closed conn $\mathbb{H}^3$ geom mfd is covered by a surface bundle as above.
Rmk: Work of Lee Mosher, Tao Li on cubulated 3-mfds.

(IV) Seifert fibered spaces:

Definition: Suppose M is conn, cpt ($\partial M \neq \emptyset$ allowed).
A Seifert fibered space structure on M is a one-dim'l foliation $\mathcal{F}$ where all leaves $L \in \mathcal{F}$ are circles.
Warning: $\mathcal{F}$ need not be unique when it exists!
Examples: (1) The Hopf fibration $S^1 \to S^3 \to S^2$

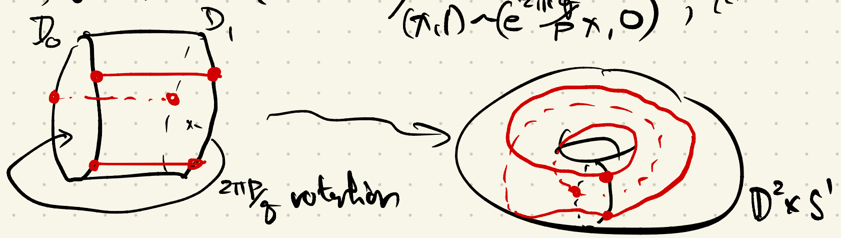
$$\overset{\substack{SO(2) \\ \|}}{S^1} \to \overset{\substack{SU(2) \\ \|}}{S^3} \to \overset{\substack{\mathbb{CP}^1 \\ \|}}{S^2}$$

Think: $S^3 \subset \mathbb{C}^2$ and the fibers are intersections of S^3 with "lines" in $\mathbb{C}^2$. **Exercise**: Fix conventions and draw the Hopf fibration in $\mathbb{R}^3$
[Is it right handed or left handed?]
(2) Fix a Riem metric on a surface F then

$$S^1 \to UT(F) \to F \text{ is a SFS str.}$$

This still works even if F is an orbifold.
Exercise: Suppose $(M, \mathcal{F})$ is SFS. Fix L a leaf. Prove that there are $p, q \in \mathbb{Z}$ ($\gcd(p,q) = 1$, $p \geq q \geq 1$) and a foliated neighbourhood $V \subset M$ of L so that

$$(V, \mathcal{F}|V) \cong \left(D^2 \times I \big/ {(x,1) \sim (e^{2\pi i \frac{q}{p}} x, 0)}, \{\{pt\} \times I\} \right)$$



Corollary: $(M, \mathcal{F})/S^1$ is a 2-dim'l orbifold.

Remark: A SFS $(M,\mathcal{F})$ is determined by the base orbifold $B = (M,\mathcal{F})/S^1$, the Seif. invariants (p_i, q_i) about crit. fibers (i.e. when $p \neq 1$) and the euler class.

Exercise: Fix $(M,\mathcal{F})$ a SFS. Fix L a leaf.

(1) prove p is determined by L (as is $q \in \mathbb{Z}/p\mathbb{Z}$)

(2) prove $p = q = 1$ for all but finitely many fibers.

(X) $\widetilde{PSL(2,\mathbb{R})}$-geometry: Fix F cpt conn closed surface with constant curv $(+1, 0, -1)$. Then

$$UTF \text{ has } \left\{ \begin{matrix} S^3 \\ \mathbb{E}^3 \\ \widetilde{PSL} \end{matrix} \right\} \text{ geom as } \chi(F) \left\{ \begin{matrix} >0 \\ =0 \\ <0 \end{matrix} \right\}$$

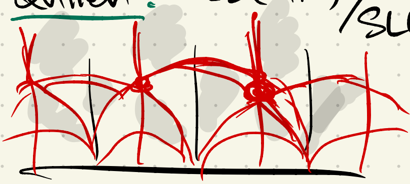
Exercises: $PSL(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2) \cong UT\mathbb{H}^2$

$\cong \mathring{D}^2 \times S^1$ [open solid torus]

Rmk: $\widetilde{PSL(2,\mathbb{R})} \cong \mathbb{R}^3$

Examples: Suppose $T \subset \mathbb{H}^2$ is a regular tessellation. Let Δ be the symmetries of T. Then $PSL(2,\mathbb{R})/\Delta$ is a $\widetilde{PSL}$ manifold.

Quillen: $SL(2,\mathbb{R})/SL(2,\mathbb{Z}) \cong S^3 -$  trefoil



$SL(2,\mathbb{Z}) \cong \Delta_{2,3,\infty}$ triangle group

$\cong$ sym of modular tiling.

(VI) $\mathbb{H}^3$-geometry:

Example [Riley, Thurston] $F =$  S^1, $f : F \to F$ the "punctured" $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ map then $M_f \cong S^3 -$ 

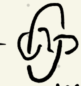
fig. 8 knot.

Riley: finds reps $\pi_1(S^3 - K) \to \mathrm{Isom}^+(\mathbb{H}^3)$
$\cong PSL(2,\mathbb{C})$

Exercise: $\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL(2,\mathbb{C})$
$\mathbb{H}^3 \cong SO(3) \backslash PSL(2,\mathbb{C})$

Thm [Riley]: The fig 8 complement is a 12-fold cover of $\mathbb{H}^3 / PSL(2, \mathbb{Z}[\omega])$

$\omega^2 + \omega + 1 = 0$

Example: $\mathbb{H}^3 / PSL(2, \mathbb{Z}[i])$ is covered by $S^3 -$  whitehead link.

Question: What is a Thurston geometry?

Answer: Fix X a 3-dimensional manifold with Riem metric let $G \cong \mathrm{Isom}(X)$. We call (G, X) a <u>thurston geom</u> If (1) G acts transitively

(2) there is some $\Gamma < G$ acting "nicely" with X/Γ compact. $\leftarrow$

(3) $\pi_1(X) = 1$.

Def: X is <u>homogen</u> iff (1)

X is <u>isotropic</u> iff $G_x = \mathrm{Stab}_x$ contains a copy of $SO(3)$.

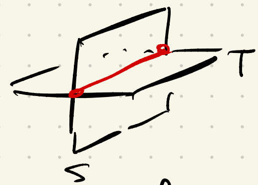
Question: what are the "thur. geom" in dim 4?

Answer: ??

Question: Do these geometries "suffice" or are they "all"?

Answer: They are all.

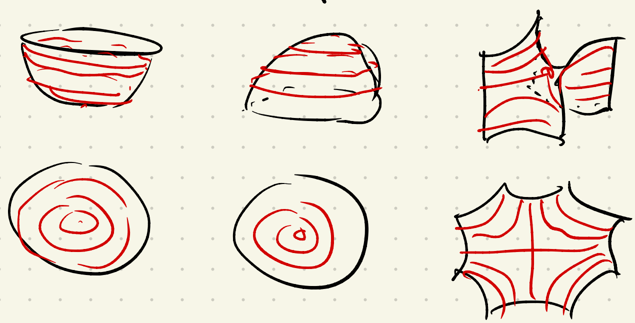
VII Transversality: A pair of transverse emb. surfaces in a 3-mfd intersect in one-manifold



Refs: Chapter 1 of Hempel [PL]
Chapter 2 of Hirsch [C^∞]

More over: A surface S transverse to a two-dim'l foliation $\mathcal{F}$ is transverse to the leaves except at finitely many points which are modelled on the following ["isolated critical pts"]

Pictures



side view

birds eye

cup +1 | cap +1 | saddle −1 | name / χ.

VIII Alexander's Theorem [Jordan-Schönflies Thm]

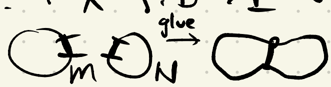
Thm: Suppose $S \subset S^3$ is a (PL or smooth) embedded two sphere. Then S separates S^3, with say $S^3 - S = D \sqcup E$ and $D \cup S \cong E \cup S \cong B^3$.

Rmks:
- $n=2$ this is true in TOP category
- $n=3$ " " false " " "
- $n=4$ " " open in PL category
- $n\geq 5$ " " true in PL category

$S^{n-1} \subset S^n$

Question: what about Brown's theorem? Answer: Need PL 4-balls.

Def: Suppose M, N are mfd with boundary. Suppose $D \subset \partial M$, $E \subset \partial N$ are emb. disks. Fix $\varphi: D \to E$ ori rev.
Define $M \#_\partial N = M \sqcup N / \varphi$.



Proposition 2.1: $M^3 \#_{\partial} B^3 \cong M^3$

Hint: Use a collar $\partial M \times I \subset M$. //