Intro to three-manifolds lecture four 2021-02-10

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Office hours in
Warwick Teams
Fridays 12:30 – 13:30

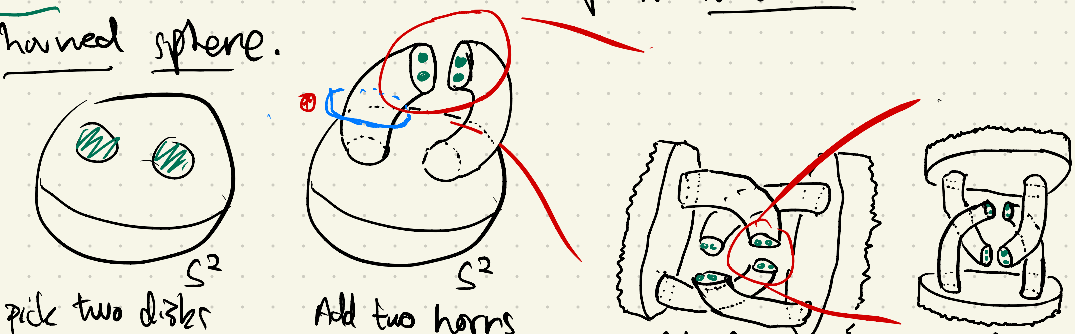
Last time: Geometries
Surface bundles
Seifert fibred spaces
Transversality.

(I) Schönflies Conjecture: Does a CAT S^{n-1} emb in S^n bound CAT B^n on each side?

CAT \ n	1	2	3	4	≥ 5
Top	✓	JCT	X	X	X
locally flat	✓	JCT	✓	✓	✓ ← Brown Mazur
PL	✓	JCT	Alex	$\boxed{\text{open}}$	h-cobord } Smale
Smooth	✓	JCT	Alex	$\boxed{\text{open}}$	h-cobord [plus Poincaré in $n=5$]

Question What is the counterexample in $n=3$, Top?

Answer: Recursive construction of the Alexander horned sphere.



pick two disks — Add two horns, pick four disks — Add four horns, pick 8 disks — and so on.

⊛ The blue loop is null homotopic in complement of any finite stage but not in complement of the limit.

(II) Collars: Manifold M with boundary ∂M has a collar $N \cong \partial M \times I \subset M$ is correct way

use this to prove

Prop 2.2: $M^3 \#_\partial B^3 \cong M^3$

(III) Isotopy and ambient isotopy: Suppose $X \subset Y$ is a subspace. A map $f: X \times I \to Y$ is called an isotopy if (define $f_t : X \to Y$ by $f_t(x) = f(x,t)$)

⊛ for all t, f_t is an embedding and ⊛ $f_0 : X \to Y$ is the inclusion

⊛: f_t homeo

Picture:

← an isotopy rel end points

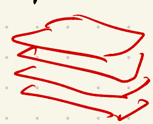
Definition: A isotopy ⊛ $f: Y \times I \to Y$ is called an ambient isotopy of X when applied to $X \subset Y$

(IV) Alexander's Theorem: Suppose $S \subset S^3$ is a (PL or smooth) emb. two sphere. Then $S^3 - S$ has two components D, E and

$$(D \cup S, S) \cong (E \cup S, S) \cong (B^3, S^2).$$

Equiv: There is an amb. isotopy of S to the round S^2.

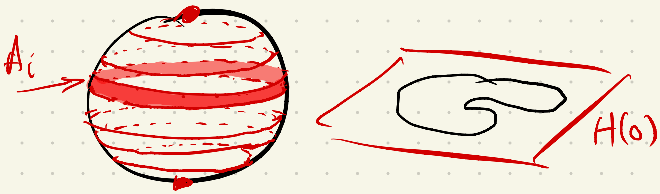
Proof: Pick $x \in S^3$, off of S. $S^3 - \{x\} \cong \mathbb{R}^3$. Fix $\mathcal{F}$ a foliation of $\mathbb{R}^3$ say by horiz. planes



Move (isotope) S to be transverse to $\mathcal{F}$.

We now use a double induction. The first measure of complexity is n = num. of saddles of S wrt $\mathcal{F}$

$\boxed{n=0}$ Then there is exactly one max and one min.

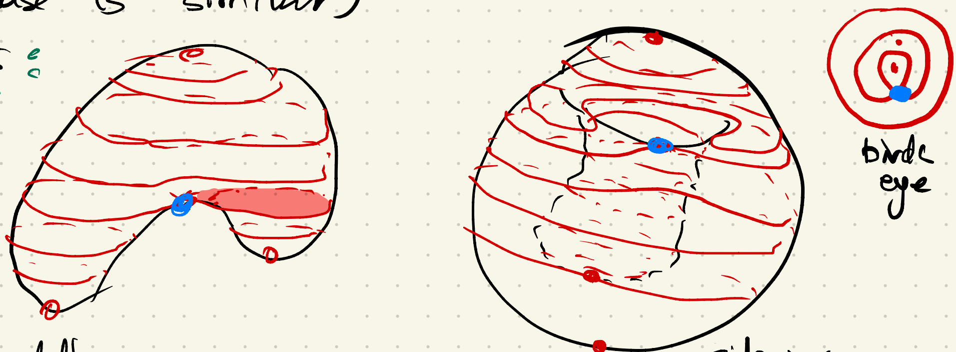


cut S into small annuli (and two disks) using $H(t_i)$ = plane at height t_i and apply JCT [and work] to annulus A_i between $H(t_i)$, $H(t_{i+1})$

Exercise: A_i bounds $D_i \cong D^2 \times I$. Apply Prop 2.2.

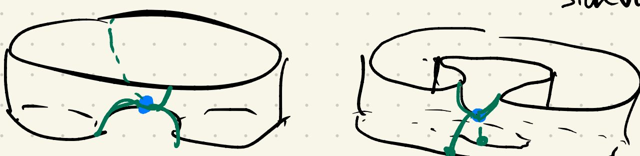
$\boxed{n=1}$ Suppose there is one max and two mins (other case is similar)

Pictures:



side view

Near the saddle

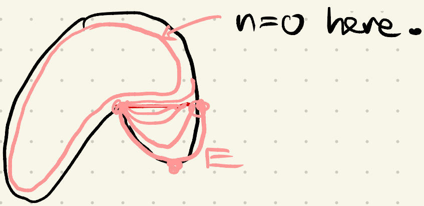


In both cases there is a disk D in the level $H(c)$ of the saddle that, with a disk E of S bounds a three ball B by the $n=0$ case.

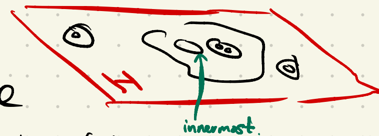
Isotope E across B to cancel a min. with the saddle. Thus the new position of S bounds a ball ✓

Cartoon

(one dim. down)



$\boxed{n \geq 2}$ Pick any level $H = H(t)$ which is regular (misses crit points) and has ≥ 1 saddle above and below. By transversality $S \cap H$ is a collection of m embedded, disjoint closed curves.

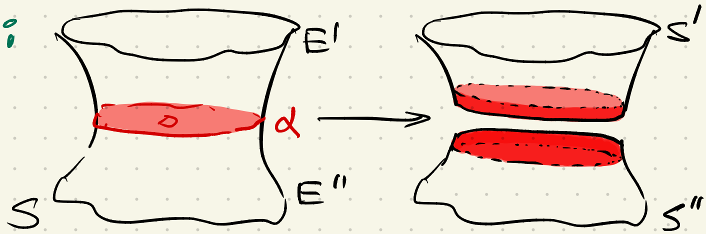


We take $m = \#$ of curves to be the second measure of complexity. [m finite b/c S cpt and $S \cap H$ transv]

$\boxed{m = 0}$ is impossible as S is conn. and H separates.

$\boxed{m \geq 1}$ Pick $\alpha \subset S \cap H$ an innermost [IVT] component of $S \cap H$ (innermost in H). Let $D \subset H$ be the disk bounded by α. Let E', E'' be the disks of $S - \alpha$. Set $S' = D \cup E'$, $S'' = D \cup E''$

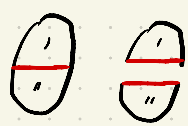
Cartoon:



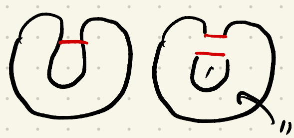
Note that in total $S' \cup S''$ has one new max. and one new min. [This procedure is called surgery or compression of S along D].

Case 1: Both S' and S'' have saddles. Induct and find three-balls B' and B'' bounded by S', S''.

Case 1a: $B' \cap B'' = \emptyset$. Then [Prop 2.2] $B' \cup (D \times I) \cup B'' \cong B^3$ and we are done.



Case 1b: $B' \subset B''$. Then [Prop 2.2] $B'' - (B' \cup (D \times I))$ is a three ball. Done.



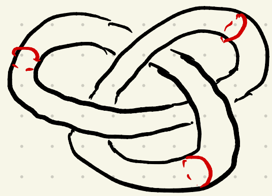
Case 1c: $B'' \subset B'$ similar.

Case 2: Suppose S' has no saddles. By $n=0$ case S' bounds. Isotope E' across 3-ball and past D. [as in $n=1$ case]. So we reduce m by one.

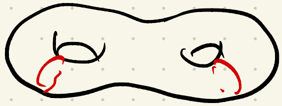
Case 3: Suppose S'' has no saddles... //Alex.

Exercise: Prove the following theorem, also due to Alexander.

Theorem 2.3: Suppose $T \subset S^3$ is a (PL or smooth) emb two-torus. Then T bounds a solid torus $(D^2 \times S^1)$ on at least one side.



Exercise: what about genus two (PL or smooth)?

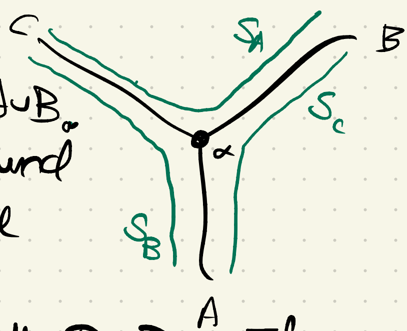


V Irreducibility:

Lemma: Suppose M is 3-mfd. Suppose $\alpha \subset M$ is an emb. loop. Suppose A, B, C are disks in M s.t. $A \cap B = B \cap C = C \cap A = \alpha$

Set $S_A = B \cup C$, $S_B = C \cup A$, $S_C = A \cup B$.

If any two of S_A, S_B, S_C bound three balls then so does the third.



Proof: Suppose S_A, S_B bound balls D_A, D_B. If $D_A \cap D_B = C$ then $D_A \cup D_B$ is a three ball [prop 2.2]

Suppose $D_B \subset D_A$. Thus $S_c \subset D_A$ and so, by Alex Thm bounds a three ball. If $D_A \subset D_B$ the proof is similar. //

Definition: Call M^3 irreducible if every $S^2 \subset M^3$ bounds a three ball.

Examples: $S^3, B^3, \mathbb{R}^3$.

Exercise: Suppose $\hat{M} \to M$ is the univ cover. If $\hat{M}$ is irred then so is M. Deduce T^3 is irred.

Lemma: $M \# S^3 \cong M$. Pf: Alex thm and Alexander Trick. //

Alex Trick: Suppose $f: B^3 \to B^3$ is homeo and $f|\partial B^3 = Id$. Then f is isotopic to Id. [Ref: Thurston] in smooth setting

Definition: Call M prime if $M \cong N \# P$ implies $N \cong S^3$ or $P \cong S^3$.

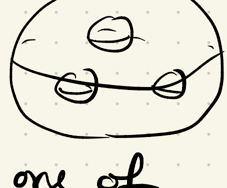
Lemma:

(1) Irred $\Rightarrow$ prime

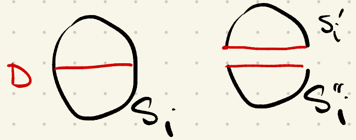
(2) Prime $\Rightarrow$ irred (or $M \cong S^2 \times S^1$ or $M \cong S^2 \tilde{\times} S^1$)

VI Independent sphere systems

Definition: M three-mfd. $S = \sqcup S_i$ a collection of emb. spheres in M. Call S indep. if no component of $M - S$ is a "punctured three-sphere" (ie. $S^3 - \sqcup B^3$'s)



Lemma: Suppose $S \subset M$ is indep. Suppose D is a disk in M s.t. $D \cap S = \partial D$. Then one of S' or S'' (obtained by comp. S along D) is indep.



$S' = (S - S_i) \cup S_i'$ [Pf: Apply 2.4]

$S'' = (S - S_i) \cup S_i''$

Question: In dim 4 is there a prime decomp. theorem?

Answer: Well probably first we need the Schönflies conjecture [in locally flat category we need the surgery theory...].

Question: How does compactness plus transversality produce finiteness?

Answer: Transversality implies the intersections (for reg leaves) are submanifolds. Now use compactness.

[Hirsch's book. Diff Topology]

Question: Do people care about analytic mfds?

Answer: Yes! Hodge conjecture!

Question: Another definition of Lens spaces?