Intro to 3-mfds [2021-02-17, lecture 5, Saul Schleimer]

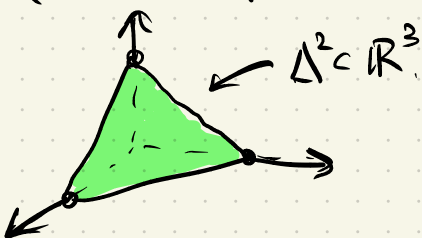
Last time:
- Isotopy
- Alexander's theorem
- Irreducibility, sphere systems.

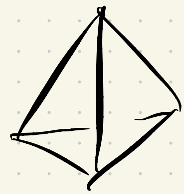
(I) Triangulations

Def: The standard n-simplex

$$\Delta^n = \{ x \in \mathbb{R}^{n+1} \mid x_i \geq 0 \text{ and } \textstyle\sum_i x_i = 1 \}$$

Picture:





Δ^3 is a tetrahedron

Def: Suppose Δ, Δ' are model simplices (copies of Δ^n). Suppose f, f' are faces (dim $n-1$). A face pairing $\phi : f \to f'$ is an isometry. [Convention: $f \neq f'$]

Def: An n-dim'l triangulation $T = \{\Delta_i, \phi_j\}$ is a collection of model simplices and face pairings s.t.

(*) If ϕ is a face pairing so is ϕ^{-1}!

Example: The boundary of Δ^n, is a triang. of S^{n-1}

Def: The realization of T is the space

$$|T| = \coprod \Delta_i \Big/ \langle \phi_j \rangle = \text{disjoint union mod equiv gen by } \phi_j$$

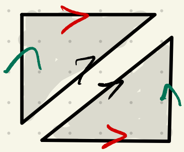
Notation: We write (M, T) to say T is a triangulation of M ($|T| \cong M$).

Notation: Let $T^{(k)}$ be the k-skeleton of T: the image in $|T|$ of the union of the model k-dim'l simplices

Notation: If f is a model facet let $\chi_f : f \to |T|$ be the characteristic map

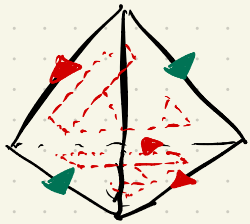
Examples: (1) $|\partial \Delta^n| \cong S^{n-1}$

(2)



$\cong T^2$

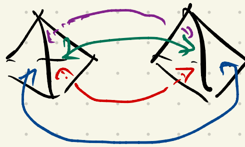
(3)



Exercise

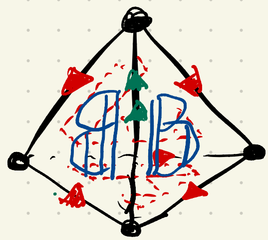
$\cong S^1 \times D^2$

(5) The double of a tetrahedron



$\cong S^3$

(4)



Exercise

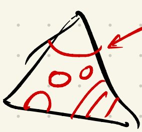
$\cong S^3$.

Exercise: In example (4) draw $T^{(1)}$ in $\mathbb{R}^3 \cong S^3 - \{pt\}$

Definition: Fix (M, T). An isotopy $F: M \times I \to M$ is skeletal if for all k, t we have $F_t(T^{(k)}) = T^{(k)}$ [For example $F_t|T^{(0)} = Id$].

Def: Suppose $S \subset M$ is prop embedded [M cpt]. Suppose T is a triang of M. We say S is transverse to T if it is transverse to $T^{(k)}$ for all k. Conclude: If f is a model face then $\chi_f^{-1}(S) \subset f$ is a k-1 dim'l submanifold of f.

Cartoon:



$\chi_f^{-1}(S)$

f

Def:

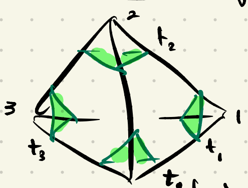
normal arc

loop

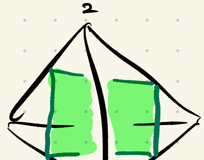
bent arc

(II) Normal surfaces:

Definition: Suppose Δ is a tet. suppose $D \subset \Delta$ is a prop. emb disk, transverse to the skeleton. We call D **normal** if it is skel. isotopic to a linear disk.



triangles.



quad

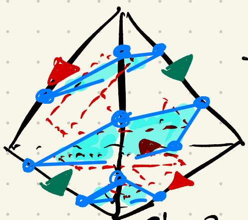
There are four/three types of tri/quad

Def: A surface $S \subset (M,T)$ is **normal** if

(1) $S \subset M$ is prop. emb.,

(2) S transverse to T, and

(3) for every $\Delta \in T$ we have $\chi_\Delta^{-1}(S)$ is a finite coll. of normal disks.



$S^1 \times D^2$

Ex: This blue surf S is a merid. disk.

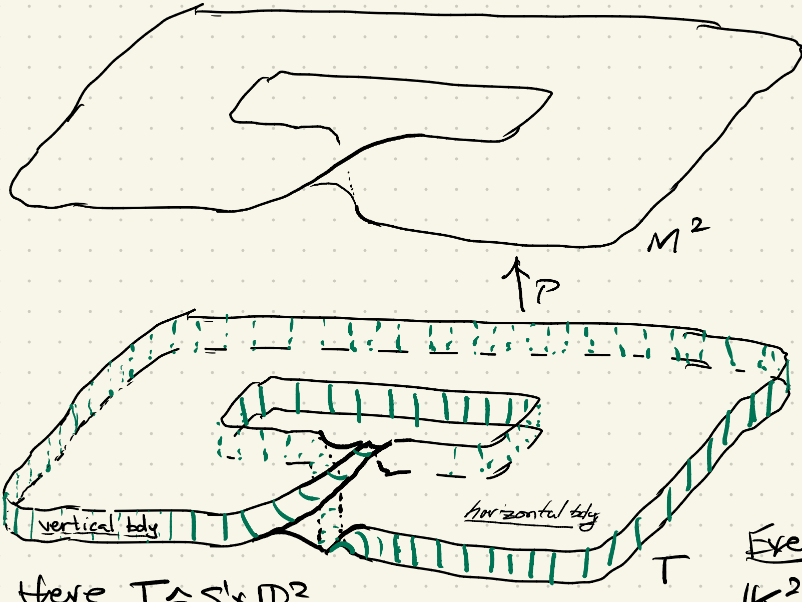
Ex: the green surf S is a two-sphere [vertex link]

(III) I-bundles:

Suppose F is a surface and $\rho : \pi_1(F) \to \mathbb{Z}_2 = \{\pm 1\}$ is a homomorphism. We build the three manifold T_ρ: this is the I-bundle over F with monodromy ρ.

$$\begin{array}{ccc} I & \longrightarrow & T_\rho \\ & & \downarrow p \\ & & F \end{array}$$

That is, if $\alpha \subset F$ is a simple closed curve then $p^{-1}(\alpha) \cong A^2$ (annulus) or M^2 (Möbius) as $\rho(\alpha) = +1$ or -1 (respectively)



Here $T \cong S^1 \times D^2$.

green is the fibers (intervals)

Exercise: Embed $K^2 - D^2$ in $\mathbb{R}^3$ and draw its orientation I-bundle

Def: If F is a surf define $p_{ori} : \pi_1(F) \to \{\pm 1\}$ by $p_{ori}(\alpha) = +1$ iff α is orient. preserving. Define $T = T_{p_{ori}}$ to be the orientation I-bundle.

Exercise: ① Suppose T is the ori. I-bund over P^2. What is this manifold? What is its boundary? What is the double of T across ∂T? [$P^3 \# P^3$]

Exercise: As above with T the ori. I-bund over K^2. Also what geometries can you get by gluing two copies of T?

Question: what is the corresp. between homo. of $\pi_1(F)$ and I-bundles over F?

Answer: I-bundles (up to bundle isomorphism) are in bijection with homo $\rho : \pi_1(F) \to \{\pm 1\}$.

Question: What is known about "minimal triangulations" [fewest tetrahedra]

Answer: 0 tets gives ϕ

k tets gives finitely many 3-mfds

[this is called Matveev complexity $c(M)$]

These are known up to $c \leq 8$ or so

Exercise: Find all M^3 that have $c(M) = \min$ tet number $= 1$

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open: Fix p, q. Find $c(L(p,q))$ [Jaco-Rubinstein-Tillmann-Speer]

[conj: $c(L(p,q)) =$ continued fraction length of q/p (-3).]

Question: what is the "order" of structures on manifolds? [and many more!]

Answer: Top $\supset$ PL $\supset C^r \supset C^\infty \supset C^\omega \supset$ Alg.

(C^ω = analytic)

Question: what is the relation between triangulable and PL?

Anwser: Most people say these are the same!

But if you are being careful: PL means has a triangulation where all vertex links are PL $n-1$ spheres.

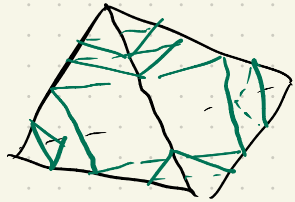
III Haken-Kneser finiteness

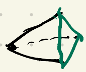
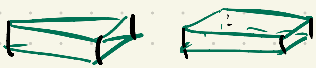
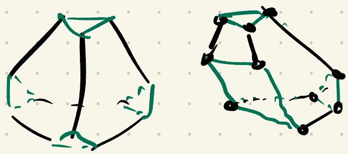
Theorem: Suppose (M, T) is a triang. 3-mfd. Suppose $t = \#$ of model tetrahedra. Suppose

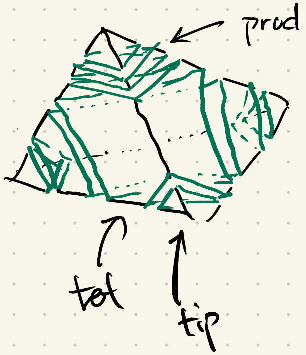
$S \subset (M,T)$ is normal. Then S has $\leq 20t$ distinct skeletal isotopy classes of connected components.

<u>Proof</u>: For each $v \in T^{(0)}$ add a small copy of the vertex link about v to S.

Fix Δ a tetrahedron. The disks of $\chi_\Delta^{-1}(S)$ include at least one of each type of triangle. Cut (M,T) along S and thus cut Δ along $\chi_\Delta^{-1}(S)$. The result in Δ is a collection of <u>blocks</u>:



- tips 
- products 
- tet blocks 



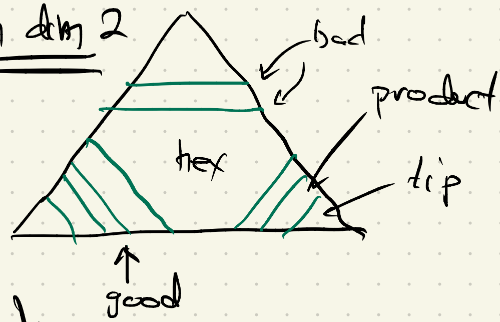
Note that Δ (after cutting) has some product blocks, 4 tips and one or two tet blocks.

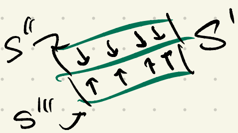
Call a normal disk of $\Delta - n(\chi_\Delta^{-1}(S))$ <u>bad</u> if it meets a non-product block. There are at most 10 bad faces in Δ (after cutting). So there are at most $10t$ bad faces in total.

Say a component $S' \subset S$ is <u>good</u> if S' only

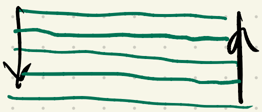
meets product blocks.
So S contains at most $10t$ bad components.
Suppose $S' \subset S$ is a good component. Define $N(S')$ to be the union of all product blocks meeting S'. $N(S')$ is an I-bundle over S'.

In dim 2



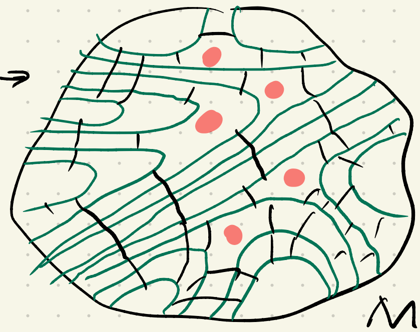


Note $S'' = S'''$ iff $N(S')$ is twisted. If not twisted all S', S'', S''' are skeletally isotopic. Finally we form the union of all $N(S')$ (for $S' \subset S$) this is again an I-bundle. There are at most $10t$ comp'ts of this union.



Each component of this I-bundle contains at most 2 skeletal isotopy classes. //

Cartoon: of S in M $\longrightarrow$



Question: How can we build non-triang manifolds?
Answer: See Cannon and Edwards papers on the double suspension theorem, and trace citations.....

Consider $PHS^3 = P \cong S^3/D^*$
Note P is a $\mathbb{Z}HS^3$.
Take the susp. of P, twice.