Intro to 3-mfds: [lecture 6 2021-02-24 Saul Schleimer]

Last time:
- Triangulations
- Normal surfaces
- I-bundles
- Haken-Kneser finiteness. (*)

(*) **Remark** HK finiteness gives an upper bound but does not give examples (at least not obviously)

Sphere Theorem: Suppose $\pi_2(M) \neq 1$. Then, for any triang K of M, (M, K) contains a emb. normal essential two-sphere.

(I) Existence of connect sum decomp:

Theorem: Fix M (cpt, conn) three-mfd. Then there is a constant $k(M)$ so that if S is an indep sys. of two spheres then $|S| \leq k(M)$.

Proof: Fix K a triangulation of M with t tetrahedra. If S is indep. then there is some S' normal indep system with $|S'| = |S|$.

Thus $k(M) \leq 20t$ by HK finiteness.

Fix the indep sys S. Isotope S to be transverse to K. No component of S lies inside any Δ^3. Here are our complexities (in order of importance)

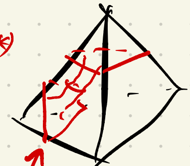
(1) $wt(S)$ = weight of S = $|S \cap K^{(1)}|$

(2) $lp(S)$ = loops of S in faces of $K^{(2)}$

(3) $nd(S)$ = non-disks of S

= sum of (neg. Euler char $(+1)$) of non-disk components of $\chi_\Delta^{-1}(S)^{(*)}$

(*) Better def: number of boundary components minus one. This has correct sub-additivity.



Planar subsurf of S.

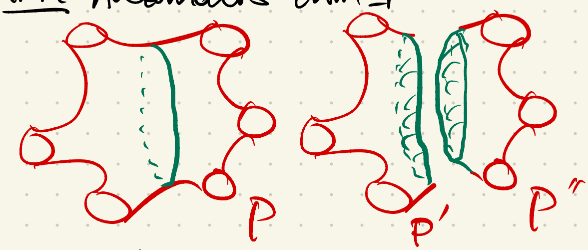
+1   +2   etc.

Note: If $lp(S)$ or $nd(S) > 0$ then S is not normal.

(3) Suppose $nd(S) > 0$. Let $P \subset \chi_\Delta^{-1}(S)$ be the offending component.

Exercise: There is a disk surgery of P in Δ. This reduces $nd(S)$ [Hint: Alexander's thm]

$-\chi = 4$, $\frac{+1}{5}$ $\longrightarrow$ $\frac{3+1}{4}$



$-\chi = 4$, $\frac{+1}{5}$ $\longrightarrow$ $-\chi = 1$, $\frac{+1}{2}$, $\frac{1+1}{2}$

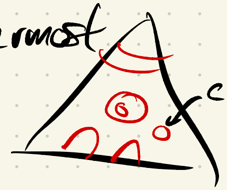
so nd decreases!

Question: why doesn't wt or lp increase?

Answer: The surgery is inside Δ.

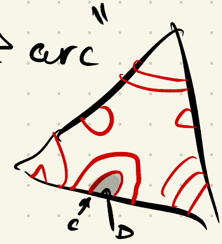
Note: by Lemma 2.6 after disk surgery and deleting a sphere (if necessary) we have a new system S'. Thus we may assume $\chi_\Delta^{-1}(S)$ is a coll. of disks.

(2) Suppose $lp(S) > 0$. Pick an innermost loop $C \subset \Delta^2$ some face of K.

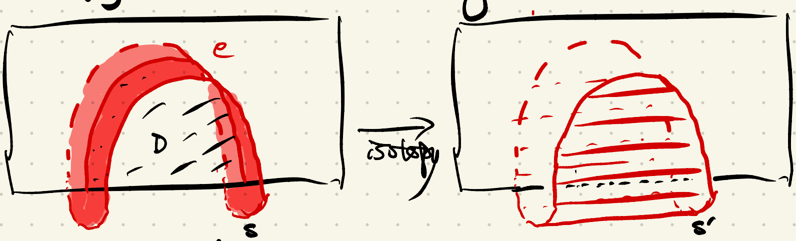


So C bounds a surgery disk $D \subset \Delta^2$ and done as above.

(1) Suppose some face Δ^2 has a "bent arc". Let e be an outermost bent arc. Let D be the disk (bigon) cut out of Δ^2 by e. We perform an ambient isotopy on S, moving e across D.



Picture



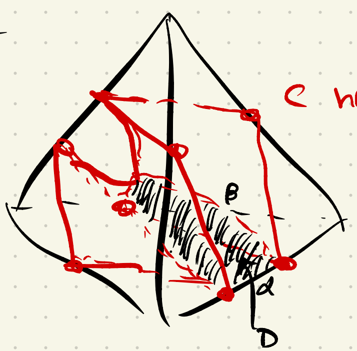
"Earmuffs" "Taco shell"

$wt(S') = wt(S) - 2$.

This gets rid of all bent arcs. Finally suppose that some component C of $S \cap \partial\Delta^3$ has length at least five.

↑ that is look at $\chi_\Delta^{-1}(S) \cap \partial\Delta$.

Picture



C normal curve.

Exercise: Thus C meets some edge of Δ^3 at least twice. This gives a baseball move as above.

Conclude: we find a indep sys. S' with $|S'|=|S|$ and S' has no long normal curves, no bent arcs, no loops, no non-disks. Thus S' is normal. //

Corollary: Suppose M cpt conn. Then M has a decomp $M \cong \#_i M_i$ as a finite conn. sum of prime manifolds.

Question: But disk surgery increases the # of spheres by one?

Answer: Yes, so we throw one of them away.

(II) Uniqueness:

Theorem 3.4: Suppose M is closed, conn, oriented. Suppose $M \cong \#_{i=1}^{k} M_i \cong \#_{j=1}^{\ell} N_j$, with all M_i, N_j prime and not S^3. Then $k=\ell$ and (after reordering) $M_i \cong N_i$ for all i. (orient preserving homeo.)

Proof: Suppose all spheres in M separate (so all M_i, N_j are irred, no copies of $S^1 \times S^2$.)

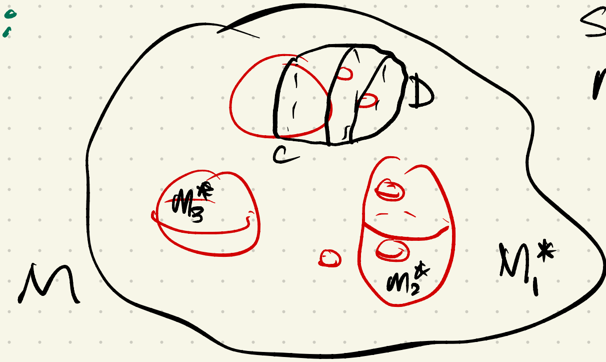
Suppose $S \subset M$ realises the first decomp $M \cong \# M_i$

Suppose $\Sigma_i \subset M$ is a two sphere cutting of $N_1^* \cong N_1 - \mathring{B}^3$

Isotope S to be transverse to Σ_i.

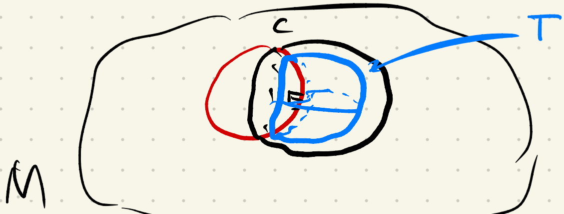
Case 1: $S \cap \Sigma_i \neq \emptyset$. So pick $c \subset S \cap \Sigma_i$ innermost in Σ_i. Let $D \subset \Sigma_i$ be the disk bounded by c.

Cartoon :



So D lies in M_1^* (after renumbering.) Note that c bounds a pair of disks in S.

Pick one of those say E. So $D \cup E$ bounds a punctured B^3 in M_1^*. [M_1 is irred]



Thus we may surger S along D and get new system S' that still realises $\underline{M \cong \# M_i}$ and

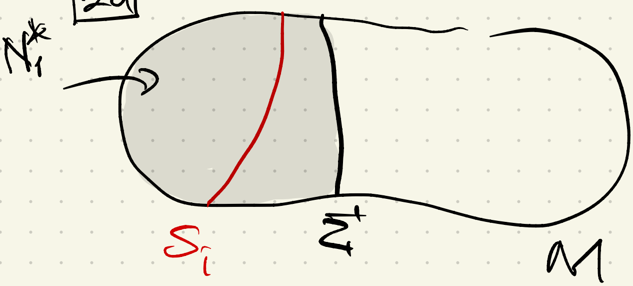
$$|S' \cap \Sigma_1| < |S \cap \Sigma_1|.$$

<u>Case 2</u>: $S \cap \Sigma_1 = \emptyset$.

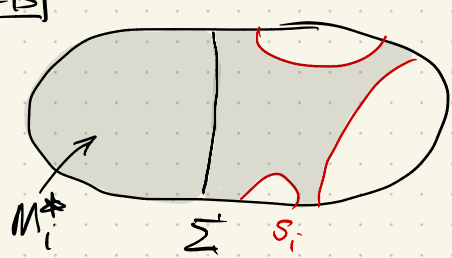
<u>Case 2a</u> Some component $S_i \subset S$ lies in N_1^*.
Since N_1 is irred, S_i is parallel to Σ_1 and some $M_i \cong N_1$ as desired.

<u>Case 2b</u>: Σ_1 lies in some M_i^*. similar.

$\boxed{2a}$



$\boxed{2b}$



Suppose now that M contains non-separating two spheres.

Let Σ_1 be a max'l system realising

$M \cong \#_j N_j$ [Question: Do we need to also cut open the $S^1 \times S^2$ factors?]

Let S be a system of r spheres so that $M - n(S)$ is conn and all spheres in $M - n(S)$ are separating.

That is:

$$M \cong \left(\#_{i=1}^{r} S^1 \times S^2 \right) \# \left(\#_{i=r+1}^{k} M_i \right)$$

We now find a sequence of disk surgeries

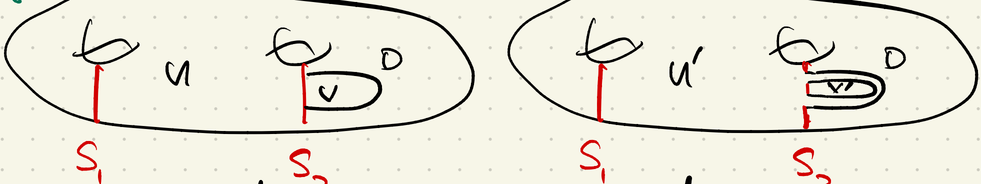
$$S \xrightarrow[D]{} S' \xrightarrow[D']{} S'' \xrightarrow[D'']{} S''' \cdots \longrightarrow S^{(n)}$$

So that $|S^{(p+1)} \cap \Sigma_1| < |S^{(p)} \cap \Sigma_1|$

We deduce that $\{N_j\}$ also has r copies of $S^1 \times S^2$.

Remains to show: $M - n(S^{(p)}) \cong M - n(S^{(p+1)})$

Picture:



Must understand how cutting along S "commutes" with surgery along D.

Define $V \sqcup V = M - (n(S \cup D))$ and think.

This finishes the proof of existance and uniqueness #.

Question: What goes wrong in dim 4?

Answer: Well the uniqueness fails... perhaps the "slides" are more dangerous than they appear?

(III) Incompressible surfaces skip!

(IV) Heegaard splittings skip!

(V) PL-minimal surfaces: The Meeks-Yau proof of the sphere theorem relies on the theory of minimal surfaces in riem. 3-mfds. Here the existance results are delicate. Instead we'll use a "combinatorial" version where existance is a simple compactness argument.

Totally general set up: we'll simplify our lives by assuming all char. maps are embeddings.

$$\begin{tikzcd} \boxed{?} \arrow[r] \arrow[d] & \Delta^3 \arrow[d, "\chi_\Delta"] \\ S \arrow[r, "f"'] & M \end{tikzcd}$$

Technical discussion. Avoid!

We now produce a thy of normal maps.

Fix K a triangulation with χ_Δ emb for all Δ.

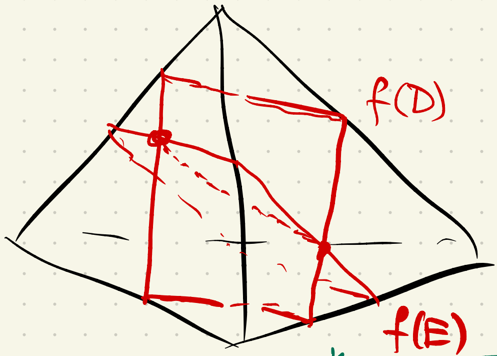
Suppose $f: S \to M$ is a map. Homotope f to be transverse to $K^{(k)}$ for all k.

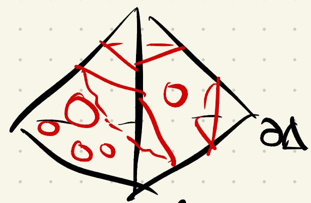
Define: $wt(f) = |f^{-1}(K^{(1)})|$

Lemma 5.1: M, K, f, S as above. After homotopy we may assume

(1) f is least weight in homotopy class.

(2) for each tet Δ in K we have, for each component c of $f^{-1}(\partial\Delta)$, either $f(c)$ is a loop in a face or $f(c)$ is a normal curve of length ≤ 4.





Note that after Lemma 5.1 we still do not know f is an embedding.

Question: Is there a sphere theorem in higher dimensions?

Answer: In dim $n \geq 5$, a generic map of a surface $S \longrightarrow M^n$ is an embedding.

So indeed elts of $\pi_2(M)$ can be "cleaned up"

Question: No, no, we care about $\pi_{n-1}(M^n)$.

Answer: The 3-dim'l proof will not generalise ...

I'll guess there is no corresponding statement.

Idea: Take N to be an $n-1$ dim'l spherical space form ($N \cong S^{n-1}/\Gamma$) and define $M = N \times S^1$ [for us, consider $P^2 \times S^1$].

Better: Set $X = S^2 \times S^2$. Show X irreducible. But $\pi_3(X)$ is non-trivial due to the Hopf map. Exercise