Introduction to three-manifolds

2021-03-17
Lecture 9
Saul Schleimer

Last time:
- Existence of PL minimal surfaces
- Balance condition at vertices
- Begin proof of disjointness.

Question: What is a reference for the problem

$$\{ \rho : \pi_1(F) \to \mathbb{Z}/2\mathbb{Z} \} \cong \{ I\text{-bundles over } F \text{ up to bundle isom} \} \, ?$$

Answer: will post ref to this kind of problem on webpage.

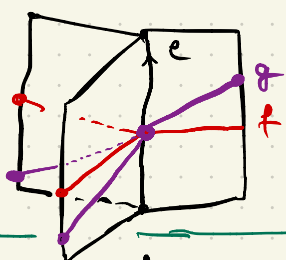
(I) Fix statement of Coro. to Lem 5.4

Corollary: Suppose $f : S \to (M, K)$ and $g : T \to (M, K)$ are PL minimal. Suppose $a \in S$, $b \in T$ with $f(a) = g(b)$ on an edge e of $K^{(1)}$. Then there are open sets $U \subset \Gamma_f$, $V \subset \Gamma_g$ with $a \in U$, $b \in V$ so that either

(*) $f(U) = g(V)$ or

(*) $g(V)$ meets both "sides" of $f(U)$

Picture:

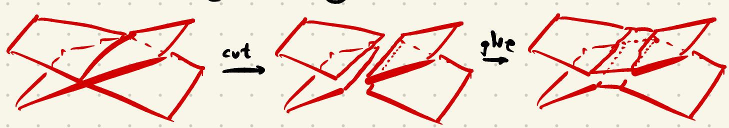


That is, we don't get control of open neigh in S, T but only in Γ_f, Γ_g

(II) Disjointness of minimal surf.

This relies on the "exchange and round off" trick and the "Meeks-Yau" trick.

Picture:



An exchange

Lemma 5.5: Suppose $f, g : S^2 \to (M, K)$ are least area and essential. Suppose both are one-to-one. Then either (i) $f(S^2) \cap g(S^2) = \phi$

(ii) $f(S^2) = g(S^2)$

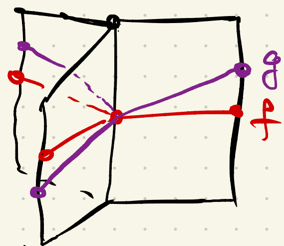
Proof: Suppose $f \cap g \neq \phi$.

Case 1: Suppose f transv. to g and $f \cap g \cap K^{(1)} = \phi$

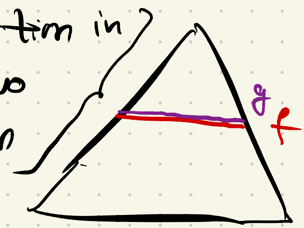
Done last time. use exchange and round off.

Case 2: Suppose not. So we see either

intersection in the one-skeleton or



intersection in the two skeleton

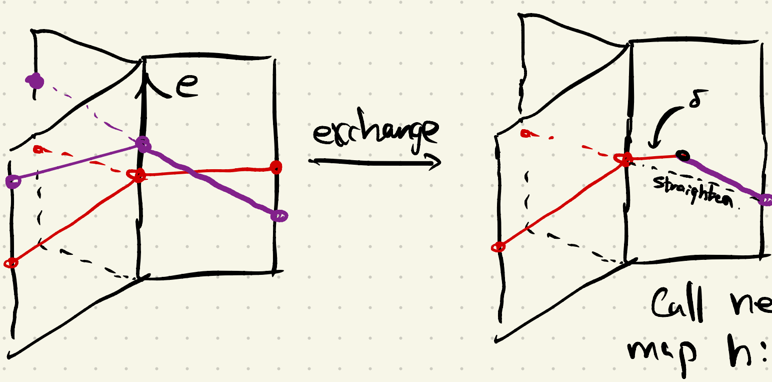


We reduce to the previous case using "MY" trick. Apply small skeletal isotopies to f and g to obtain f', g' [straight] Now f', g' are as in case 1. with

$$w(f) = w(f') \quad \text{and} \quad \ell(f') < \ell(f) + \varepsilon$$
$$w(g) = w(g') \qquad \ell(g') < \ell(g) + \varepsilon.$$

If $f' \cap g' = \phi$ then we contradict corollary [say, b/c g is locally "above" f].

Picture:

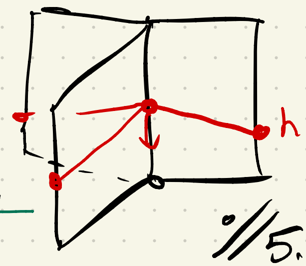


Call new straight map $h : S^2 \to (M, K)$

Note $w(h) \le w(f) = w(g)$

However we may have $\ell(h) > \ell(f) = \ell(g)$ [by at most ε]. So let φ_i be the angles of h about e. Here we have $\varphi_i = \max\{\theta_i, \phi_i\}$ [the angles of f, g] because h is (at $f(a) = g(b)$) below both f, g.

So h is not balanced and we can reduce $\ell(h)$ by a def amount $\ast$ [that is, we can slide a vertex down to reduce $\ell(h)$]



//5.5.

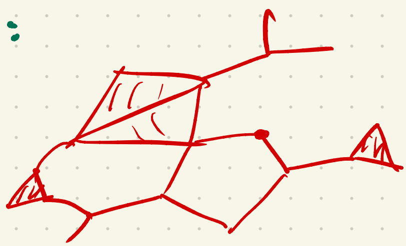
<u>Question</u>: Where do we see the conclusion $f(S^2) = g(S^2)$ of lemma?

<u>Answer</u>: Define $\Sigma^1(f) = f^{-1}(f \cap g)$

$$\Sigma^1(g) = g^{-1}(f \cap g)$$

Either $\Sigma^1(f) = S^2$ and we are done or $\Sigma^1(f)$ has a vertex on its "boundary".

<u>Picture of $\Sigma^1(f)$</u>:



<u>Exercise</u>: Give the details.

(III) The sphere (and disk, torus, annulus) theorems.

<u>Theorem</u> [Sphere] Suppose M is closed conn three mfd. Suppose M contains an essential sing. two-sphere. Then it contains an embedded ess two-sphere.

(IV) A bit of alg. topology:

<u>Lemma</u>: Suppose M^3 is cpt and oriented. Then the kernel of the induced homomorphism $i_*: H_1(\partial M) \longrightarrow H_1(M)$ has rank one half that of $H_1(\partial M)$. [This requires "three-dimensionality"]

Pf: Exercise. <u>Hint</u>: Poincaré duality. //

<u>Proposition</u>: Suppose M is cpt and $\pi_1(M) \cong 1$

Then (1) all cmpts of ∂M are two-spheres.

(2) $\pi_2(M)$ is gen by $[S]$ for $S \subset \partial M$ compt

Also

<u>Lemma</u>: Suppose M is cpt and $\pi_1(M) \cong 1$

Then $\sum_{S \subset \partial M}' [S] = 0$. Pf ∂M is zero in $H_2 \cong \pi_2$. //

<u>Examples</u>: $S^3 - \sqcup$ small three-balls has $\pi_1 \cong 1$.

(V) <u>Proof of sphere theorem</u>: Given M closed connected three manifold with $\pi_2(M) \not\cong 1$.

Fix K a triangulation of M. Fix $f: S^2 \to (M, K)$

Assume f is PL min among ess sing two-spheres.

[so f is normal, straight]

[<u>Historical aside</u>: The issue for attempts to prove this was <u>triple points</u>]

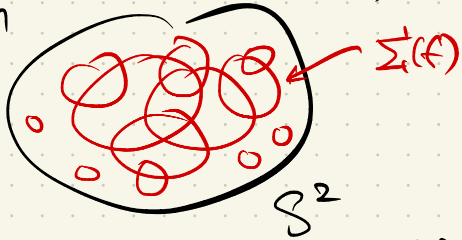
Define $\Sigma_1(f) = \{ x \in S^2 \mid$ there is some $y \in S^2$ with $x \neq y$, $f(x) = f(y) \}$

"Locus of non-injectivity"

Case 1: Suppose f is self transverse and $f(\Sigma_i^1)$ misses $K^{(1)}$.

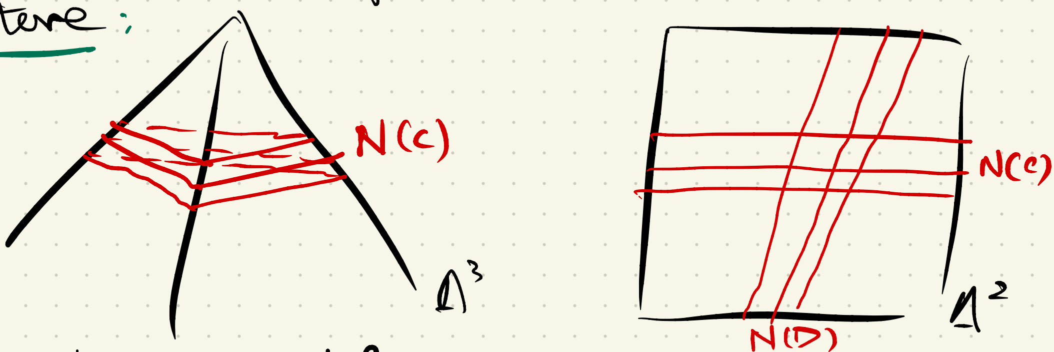
Thus Σ_i^1 is a union of transverse curves in S^2.

Cartoon



Suppose C is a comp't of $S - \Gamma_f$. So $f|C$ is a normal disk. Let $N(C)$ be an ε-neigh of $f(C)$. Define $N = \bigcup_C N(C)$.

Picture:



Note that N def retracts to $f(S)$ so $\pi_1(N) \cong \pi_1(f(S))$.

VI) The tower [Papakyriakopoulos 1957]

The base of the tower:

$$\text{Set } f_0 = f,\ N_0 = N,\ M_0 = M.$$

$$\text{So } S^2 \xrightarrow{f_0} N_0 \subset M_0$$

Climbing the tower: we are given

$$S^2 \xrightarrow{f_k} N_k \subset M_k$$

If $\pi_1(N_k) \cong 1$ set $n = k$. we are at the top.

If not: Set $M_{k+1} = \tilde{N}_k$ univ. cover.
Lift f_k to a map $\tilde{f}_k : S^2 \longrightarrow \tilde{N}_k = M_k$.
Let N_{k+1} be an (even smaller) reg neigh of $\tilde{f}_k(S^2)$ in M_{k+1}. Finally restrict the range to obtain $f_{k+1} : S^2 \longrightarrow N_{k+1}$

The tower

$$
\begin{tikzcd}
 & N_k \arrow[r, phantom, "\subset"] & M_k \arrow[r, phantom, "="] & \tilde{N}_{k-1} \arrow[dll] \\
 & N_{k-1} \arrow[r, phantom, "\subset"] & M_{k-1} \arrow[r, phantom, "="] & \tilde{N}_{k-2} \arrow[dll] \\
 & \vdots & \vdots & \vdots \arrow[dll] \\
 & N_1 \arrow[r, phantom, "\subset"] & M_1 \arrow[r, phantom, "="] & \tilde{N}_0 \arrow[dll] \\
S^2 \arrow[r, "f_0"] \arrow[ur, "f_1"] \arrow[uuur, "f_{k-1}"] \arrow[uuuur, "f_k"] & N_0 \arrow[r, phantom, "\subset"] & M_0 \arrow[r, phantom, "="] & M
\end{tikzcd}
$$

Rmk: "Projecting" f_k down gives $f_0 = f$.
[that is, composing the covers and inclusions]

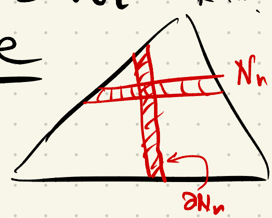
Step 1: The tower is finite: $\pi_1(N_n) \cong 1$.

Pf: Note $\Sigma_k = \Sigma_1(f_k)$ is a finite graph and $\Sigma_{1k+1} \subset \Sigma_{1k}$. So the sequence (Σ_{1k}) stabilises
[Exercise: Prove the above]
If $\Sigma_{1k+1} = \Sigma_{1k}$ then $N_{k+1} \longrightarrow N_k$ is a homotopy equivalence $\divideontimes$.
[Again Exercise]

Step 2: If f_n is one to one then $n=0$.
Pf: $G=\pi_1(N_{n-1})$ is the deck group of $M_n \longrightarrow N_{n-1}$. Pick $\gamma \in G$ non-trivial.
Define $g_n = \gamma \circ f_n$. So g_n, f_n are emb. and ~~transverse~~. Apply 6.5 and do a disk surgery if $f_n \cap g_n$ is non-empty. So get a smaller area map $h: S^2 \to M$. $\divideontimes$
Thus $f_n \cap g_n = \phi$ and this holds for all $\gamma \in G - \{1\}$. Thus f_{n-1} was one-to-one $\divideontimes$.

Step 3: f_n is one-to-one.
With f_n, N_n, M_n as above: All components of ∂N_n are two-spheres, b/c $\pi_1(N_n) \cong 1$.
Note $[f_n] \in \pi_2(N_n)$ is non-trivial [Exercise]
So: As $\{[S] \mid S \subset \partial N_n \text{ compt}\}$ generate $\pi_2(N_n)$ there are at least two nontrivial spheres in that collection.
Suppose for contradiction that f_n is not 1-1.

Picture



Note $w(\partial N_n) \leq 2w(f_n) = 2w(f)$

Also $\ell(\partial N_n) \leq 2\ell(f_n) + \varepsilon = 2\ell(f) + \varepsilon$

Straightening reduces $\ell(\partial N_n)$ by a def amount

so $\ell(\partial N_n^*) < 2\ell(f)$.

Thus, one of the (at least two) ess. cmpts of ∂N_n either has smaller weight or has same weight and smaller length. This contradicts the minimality of f.

Case 2: Either f is not self-transverse or $\Sigma_1^+(f)$ meets $K^{(1)}$. Apply Meeks-Yau: perturb f to g as in case 1 but at cost of increasing $l(f)$ by small amount. Now proceed as in case 2 of 5.5. When $\Sigma_1'(f_n) = S^2$ then f_n is a double cover of a proj. plane. //

Main ingredients:

(1) PL min. surfaces [Jaco-Rubinstein]
(2) Exchange and round-off [and MY] tricks
(3) the tower [Papakyriakopoulos]