CHAPTER 15

3-Dimensional Topology up to 1960

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1. Introduction

In this paper we discuss the development of 3-dimensional topology, from its beginnings in the 1880's, up until roughly 1960. The decision to stop at 1960 was more or less arbitrary, and indeed we will sometimes briefly describe developments beyond that date. Our account is very much in the nature of a survey of the literature (an internal history, if you will), an approach which is feasible because the literature is so finite. (This continues to be true through the 1960's, when the number of people working in 3-dimensional topology was still relatively small. During the last twenty years or so, not only has the actual literature grown tremendously, but the number of major themes in the subject has also increased.)

The early papers that deal with 3-manifolds are few: Poincaré's *Analysis Situs* [79] and his fifth complement to that paper [81], Heegaard's dissertation [44], Tietze's Habilitationschrift [110], and the paper of Dehn [22], is almost a complete list up to the end of the First World War, and one or two short papers of Alexander, together with Kneser's paper [56], then take us through the next decade. The 1930's saw an increase in activity, with the work of Reidemeister, Seifert, Seifert and Threlfall, and others, in Germany, and, in England, J.H.C. Whitehead, but this ended with the Second World War, and not much more appeared until the 1950's, when we find Moise's proof of the existence and uniqueness of triangulations [65], Papakyriakopoulos' proof of Dehn's lemma and the sphere theorem [72], and, at the end of the decade, Haken's use of normal surfaces to solve the knot triviality problem [42].

This last is an instance where it would be artificial to try to separate knot theory from 3-dimensional topology (the discussion of Dehn surgery in [22] is another), but in general we have ignored papers that deal specifically with knots, such as Dehn's 1914 paper [23]. Another topic that we do not discuss is "wild" topology.

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2. Beginnings

Possibly the first attempt at a systematic approach to the study of 3-manifolds is contained in a short note by Walther Dyck in the Report of the 1884 Meeting, held in Montreal, of the British Association for the Advancement of Science [27]. He says his goal is to classify 3-manifolds:

> The object is to determine certain characteristical numbers for closed threedimensional spaces, analogous to those introduced by Riemann in the theory of his surfaces, so that their identity shows the possibility of its 'one–one geometrical correspondence'.

He offers the following method of construction of 3-manifolds:

> We cut out of our space[1] $2k$ parts, limited by closed surfaces, each pair being respectively of [genus] $p_1, p_2, \ldots, p_k$. Then, by establishing a mutual one–one correspondence between every two surfaces, we close the space thus obtained.

The "characteristical numbers" he had in mind are the genera $p_1, p_2, \ldots, p_k$ of the surfaces and "the manner of their mutual correspondence".

Presumably Dyck's construction was suggested by the fact that any closed orientable surface can be obtained by removing an even number of disjoint disks from S^2 and identifying the resulting boundary components in pairs. Perhaps it was also this analogy that led him to make the rather rash claim that

> we can form all possible threedimensional spaces by [this] procedure.

Dyck notes that his construction gives a 3-manifold containing nonseparating surfaces, and closed curves "which can neither be transformed into each other, nor be drawn together into one point".

To illustrate his remark about closed curves, Dyck gives as examples the two 3-manifolds obtained by removing a pair of solid tori from S^3 and identifying the resulting boundaries, firstly, so that meridians are identified with meridians and latitudes with latitudes,[2] and secondly, so that meridians are identified with latitudes and vice versa. (These manifolds are, respectively, the connected sum of two copies of $S^1 \times S^2$, and $S^1 \times S^2$.) He points out that in the first case a meridian of one of the solid tori cannot be shrunk to a point, while in the second case it can.

Why was Dyck interested in 3-manifolds? He says his motivation was "... certain researches on the theory of functions, ...", and also mentions the theory of Abelian integrals. Poincaré is more explicit. In the introduction to his 1895 *Analysis Situs* paper [79], (which we discuss below), he gives three examples to justify his interest in manifolds of dimension greater than 2. (Note that when Poincaré talks about n-dimensional *Analysis Situs* he means the study of $(n-1)$-manifolds in $\mathbb{R}^n$.)

> The classification of algebraic curves into genera depends, after Riemann, on the topological classification of real closed surfaces. An immediate induction makes us understand that the classification of algebraic surfaces and the theory of their birational

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1 Dyck explains that by this he means S^3, the one point compactification of $\mathbb{R}^3$.

2 These suggestive terms for the two obvious isotopy classes of curves on the boundary of a solid torus became standard. Somewhere along the way, however, (possibly first in [121]), "latitude" mistakenly became "longitude". In this article we will revert to the original terminology.

> transformations is intimately connected with the topological classification of real closed surfaces in 5-dimensional space.
>
> Again, ..., I have used ordinary 3-dimensional *Analysis Situs* in the study of differential equations. The same researches have been pursued by Dyck. One sees easily that generalized *Analysis Situs* would allow one to treat in the same way equations of higher order, and, in particular, those of celestial mechanics.
>
> Jordan has determined analytically the groups of finite order contained in the linear group of n variables. Klein has earlier, by a geometric method of rare elegance, solved the same problem for the linear group of two variables. Could one not extend Klein's method to the group of n variables, or even to an arbitrary continuous group?

Finally, Heegaard, in the preface to his dissertation [44], explains the motivation for his investigations:

> The theory of functions with one independent variable is very closely connected with the theory of algebraic curves. The geometry of such a curve becomes therefore of fundamental importance.

He recalls that one approach to this was "the topological examinations of the Riemann surfaces that represent the algebraic curve". He goes on to say:

> The transformations of algebraic surfaces play an analogous role in the theory of functions of two variables,

but regrets that although there has been some attempt to generalize "the Riemann–Betti theory of connectivity numbers" to higher-dimensional manifolds (mentioning Picard, Poincaré and Dyck in this connection), "a completely satisfactory account is nowhere to be found". Therefore, he says, before embarking on this approach, "we need a theory of correspondence of manifolds of dimension greater than 2".

These are some of the considerations that provided the impetus for the study of manifolds of dimension greater than 2. It was only natural that the first case, of dimension 3, should receive special attention.

3. Poincaré's Analysis Situs

Three-dimensional topology was really born in Poincaré's foundational paper [79], published in 1895, (the results were announced in 1892 [77]), where we find it inextricably linked with the origins of topology in general. Paper [79] introduces manifolds, homeomorphism, homology, Poincaré duality, and the fundamental group, and in it 3-manifolds appear as examples, both to illustrate these general concepts and also with which to test the strength of the topological invariants (the Betti numbers and the fundamental group) that Poincaré has defined.

The first mention of 3-manifolds in [79] is to illustrate Poincaré's definition of the Betti numbers of a manifold V. Having defined homology in V in terms of m-submanifolds bounding $(m+1)$-submanifolds, he then explains that homologies can be combined in the same way as ordinary equations, and defines the m-th Betti number $\beta_m(V)$ to be the maximal number of linearly independent m-dimensional submanifolds of V.[3] To "clarify

[3] Actually Poincaré works with $P_m = \beta_m + 1$, but we will adopt the modern convention (which in [95] is attributed to Weyl).

these definitions", Poincaré considers a submanifold V of $\mathbb{R}^3$ bounded by n disjoint closed surfaces $S_1, \ldots, S_n$, and asserts that

$$\beta_2(V) = n - 1, \qquad \beta_1(V) = \frac{1}{2}\sum_{i=1}^{n} \beta_1(S_i),$$

mentioning as particular examples the region bounded by a sphere, the region between two spheres, the region bounded by a torus, and the region between two tori. In these formulae we see early hints of Poincaré–Lefschetz duality.

More important for 3-dimensional topology is Poincaré's description of 3-manifolds as being obtained by identifying faces of 3-dimensional polyhedra. Interestingly, Poincaré regards the 3-manifold V itself as being embedded in $\mathbb{R}^4$, but points out that, if it can be decomposed into pieces that are homeomorphic to polyhedra in $\mathbb{R}^3$, in such a way that the intersections of the pieces correspond to faces of the polyhedra, then

> ... the knowledge of the polyhedra P_i and the way their faces are identified provides us, in ordinary space, with an image of the manifold V, and this image suffices for the study of its properties from the point of view of *Analysis Situs*.

He then gives the following five explicit examples of face identifications of a single polyhedron P, four with P being the cube, and one with P an octahedron.

(1) Opposite faces of the cube are identified with no rotation, i.e. by reflection in the parallel plane midway between them.

(2) Two pairs of opposite faces of the cube are identified with an anticlockwise rotation through $\pi/2$, and the third pair with a clockwise rotation through $\pi/2$.

(3) Opposite faces of the cube are identified with an anticlockwise $\pi/2$ rotation. (There is a misprint in the identification of the second pair of faces, but this is clearly what is intended.)

(4) Two pairs of opposite faces of the cube are identified with no rotation, and the third pair with a rotation through π.

(5) Opposite faces of a regular octahedron are identified by reflection in the center of the octahedron.

Poincaré returns to these examples later, but let us note here that (1) is the 3-torus T^3, (3) is quaternionic space, (4) is the T^2-bundle over S^1 with monodromy $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, and (5) is 3-dimensional real projective space $\mathbb{RP}^3$.

Poincaré explains that a space constructed from a polyhedron P in this way will be a 3-manifold if and only if the link of every vertex is a sphere, and shows, using Euler's formula, how this can be checked from the manner of identification of the faces of P. In particular, this shows that all the above examples except (2) are indeed manifolds.

The most far-reaching discussion in [79] for 3-dimensional topology, however, begins with Poincaré considering the idea of obtaining a 3-manifold as the quotient of a properly discontinuous action of a group G on $\mathbb{R}^3$, relating this to the previous definition by pointing out that such a manifold can be described by identifying suitable faces on the boundary of a fundamental domain. He says:

> The analogy with the theory of Fuchsian groups is too obvious to labour; I will restrict myself to a single example.

Despite this remark, it seems that Poincaré was not aware of any examples of hyperbolic 3-manifolds, although he had already, in his 1883 memoir on Kleinian groups [78] (see [105] for an English translation), described the action of $PSL_2(\mathbb{C})$ on the upper half-space model of hyperbolic 3-space.

At any rate, his "single example" is in fact the infinite family of examples M_A, one for each matrix $A = \left[\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right] \in SL_2(\mathbb{Z})$, the corresponding group G_A being the group of affine transformations of $\mathbb{R}^3$ generated by:

$$\begin{aligned}(x, y, z) &\longmapsto (x+1, y, z),\\ (x, y, z) &\longmapsto (x, y+1, z),\\ (x, y, z) &\longmapsto (\alpha x + \beta y, \gamma x + \delta y, z+1).\end{aligned}$$

Thus M_A is the T^2-bundle over S^1 with monodromy induced by the linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$.

After describing these examples, Poincaré begins the next section with the sentence:

> We are thus led to the notion of the fundamental group of a manifold.

He introduces this with a discussion of how the values of a multi-valued function on a manifold V at a point may change when the point describes a loop in V. Thus the function undergoes a "substitution", the set of which, when we consider all possible loops, forms a group. He then defines the *fundamental group* of V as the group of (based) homotopy classes of loops in V, and states that a group of the first type will always be a quotient of this fundamental group.

Poincaré was very much aware of the importance of the fundamental group. In the 1882 announcement [77] of some of the results which were to appear in [79], he says:

> The group G may thus serve to define the form of the surface [4] and may be called the group of the surface. It is clear that if two surfaces can be transformed one into the other by way of continuous deformation, their groups are isomorphic. The converse, although less evident, is nonetheless true, for closed surfaces, so *that which defines a closed surface from the point of view of* Analysis situs, *is its group.*

By the time he wrote [79], this last claim had been downgraded to a question:

> It would be very interesting to treat the following questions:
>
> 1. Given a group G defined by a certain number of fundamental equivalences, can it give rise to a closed n-dimensional manifold?
> 2. How can one construct this manifold?
> 3. Are two manifolds of the same dimension, which have the same group G, always homeomorphic?
>
> These questions would require difficult studies and long developments. I will not speak of them here.

No doubt Poincaré would have been able to answer his third question if he had thought about it a little longer: examples such as S^4 and $S^2 \times S^2$ would surely have occurred to him. But he had other things to do, and, having put the question aside, he apparently did not return to it. It turns out that there are even nonhomeomorphic 3-manifolds with the same group, namely lens spaces (see Section 7). Nevertheless, in dimension 3 Poincaré's

[4] Recall that by a "surface" Poincaré means an n-dimensional manifold in $\mathbb{R}^{n+1}$.

question is very much to the point: conjecturally, any closed, irreducible 3-manifold, which is not a lens space, is determined by its fundamental group.

Poincaré shows how to derive a presentation for the fundamental group of a 3-manifold obtained by identifying faces of a polyhedron P: there is a generator ("fundamental closed path") for each pair of identified faces, namely the loop defined by joining, by a pair of arcs, a base point in the interior of P to corresponding points in the two faces, and a relation ("fundamental equivalence") for each edge in the manifold, which sets the product of the generators corresponding to the faces around that edge equal to the identity. From the fundamental group, the first Betti number may be calculated simply by abelianizing:

> When one has thus formed the fundamental equivalences, one may deduce the fundamental homologies, which differ only in that the order of the terms is immaterial. The knowledge of these homologies immediately lets one know the Betti number P_1.

Applying this to his earlier examples, he obtains the following presentations for the fundamental groups:

(1) $\langle a, b, c\colon ab = ba,\ ac = ca,\ bc = cb\rangle$; $\beta_1 = 3$.

(3) $\langle a, b, c\colon a^2 = b^2 = c^2,\ a^4 = 1,\ c = ab\rangle$; $\beta_1 = 0$.

He notes that this is a group of order 8, which acts on $\mathbb{R}^4$ (this action is quaternionic multiplication, if we identify the group with $\{\pm 1, \pm i, \pm j, \pm k\}$) so as to leave invariant the cube with faces $x_i = \pm 1$, $1 \leqslant i \leqslant 4$. For this reason he suggests that it might be called the *hypercubic group*.

(4) $\langle a, b, c\colon bc = cb,\ ca = ab,\ b^{-1}a = ac\rangle$; $\beta_1 = 1$.

(5) $\langle a\colon a^2 = 1\rangle$; $\beta_1 = 0$.

Turning to the examples M_A, Poincaré notes that here $\pi_1(M_A) \cong G_A$. He then computes the Betti numbers, showing that

$$\beta_1(M_A) = \begin{cases} 3, & \text{if } A = I, \\ 2, & \text{if trace } A = 2 \text{ and } A \neq I, \\ 1, & \text{otherwise.} \end{cases}$$

There then follows a detailed proof, by a direct group-theoretic argument, that $G_A \cong G_{A'}$ if and only if A and A' are conjugate in $GL_2(\mathbb{Z})$. (The proof distinguishes the three cases, A hyperbolic, elliptic, or parabolic. It is interesting to note that in terms of Thurston's eight 3-dimensional geometries [109], these cases correspond to M_A having a geometric structure modelled on Sol, E^3, and Nil, respectively; see [92, Theorem 5.5].) In particular, Poincaré concludes that there are infinitely many distinct closed 3-manifolds with the same Betti numbers.

Poincaré also remarks that the fundamental groups of his examples (3) and (5) are finite, of orders 8 and 2, respectively, while the group of the 3-sphere is trivial. Thus no two of these manifolds are homeomorphic, but, on the other hand, since their groups are finite, their Betti numbers are zero. In view of this, Poincaré suggests that

> It would seem natural to restrict the meaning of the term *simply connected* and to reserve it for manifolds with trivial fundamental group.

Poincaré wrote five complements to *Analysis Situs*. The first two were in response to the criticisms of [79] by Heegaard, who pointed out, among other difficulties, that Poincaré's duality theorem for the Betti numbers appeared to be false, citing as an example the manifold obtained by gluing together two solid tori in such a way that a meridian of one is identified with a curve that winds twice latitudinally and once meridionally on the other (in other words, real projective space $\mathbb{RP}^3$). Heegaard points out that every 2-cycle in $\mathbb{RP}^3$ bounds, but there is a 1-cycle which does not. The problem is, of course, torsion: Poincaré's definition of the Betti numbers "allows division" [82, Section XVI], in contrast to Betti's definition. At any rate, one of the consequences of this was that Poincaré realized that one could work with homology "without division", and obtain additional invariants, which he called *torsion coefficients*.

In the second complement, [80], Poincaré computes the torsion coefficients of the 3-dimensional manifolds described in [79]. In particular, his examples (3) (quaternionic space) and (5) (real projective space) have the same Betti numbers and torsion coefficients (i.e. the same first homology group, namely $\mathbb{Z}_2$), but have nonisomorphic fundamental groups. Curiously, Poincaré does not explicitly mention this, nor does he note that the manifolds M_A also provide examples of this phenomenon (although not so obviously).

Poincaré does not mention 3-manifolds again in [79], and in the first four complements, their only brief appearance (in the second) is the one we have just mentioned. He returns to them in a big way, however, in his fifth complement, which we discuss in Section 5.

We conclude by remarking that it has become conventional to accuse Poincaré of being obscure and sometimes lacking in rigor, but anyone who does so should reflect that things could have been worse. At the end of the introduction to [79] he says:

> ... my only regret is that [this memoir] is too long; but when I have wanted to restrain myself, I have lapsed into obscurity; I have preferred to be considered a little talkative.

4. The Heegaard diagram

Although the term *Heegaard diagram* eventually acquired a quite specific meaning, Heegaard's original definition of a "diagram" was considerably more general. This is given in his 1898 dissertation [44]. (Because of the influence this work had on Poincaré, a French translation was published in 1916 [45]. An English translation of part of the dissertation has recently been made by A.H. Przybyszewska; see [83]. For a very interesting account of Heegaard's life, see [67].)

In order to investigate the topology of manifolds of dimension greater than 2, Heegaard decides not to take as his model the Riemann–Betti theory of connectivity numbers (as Dyck, Poincaré and Picard had done), but instead to try to generalize the *puncture method* of Petersen "which I recalled from lectures", in which one "puncture[s] the Riemann surface and bring[s] it by continuous deformation into normal form". He explains further:

> The question that we first meet is this: how is one to cut a closed manifold to make it simply connected? To solve this problem we use the following procedure: the manifold is punctured, i.e. a 3-cell neighborhood of a point is removed. Thus a boundary is created, which is enlarged by a continuous deformation so as to remove more and more of the given manifold. We continue in this way until certain parts of the boundary meet others, stopping the deformation in these places when the distance between the parts that are meeting has become infinitely small. In this way we are led to a *diagram* consisting of a system of manifolds of lower dimension than the given one, or rather the neighborhood of this system, i.e. a manifold which is infinitely small in the n-th dimension. The system of lower-dimensional manifolds which constitutes the boundary of the diagram is called the *nucleus*.

Thus the nucleus is an $(n-1)$-dimensional spine of the manifold, and the diagram is a neighborhood of the nucleus, with the cell decomposition of the nucleus as part of the data. In other words, a diagram is essentially a handle decomposition.

Specializing to the 3-dimensional case, Heegaard starts with a 3-manifold obtained by identifying the faces of a polyhedron, the nucleus being the 2-complex resulting from the identifications on the boundary of the polyhedron. The diagram then consists of neighborhoods of the 0-cells, 1-cells, and 2-cells; these neighborhoods he calls *junction spheres*, *strings*, and *plates*.

The boundary of the union of the junction spheres and strings is a surface (which Heegaard allows to be non-orientable) with "connectivity number" $2p+1$,[5] say; Heegaard then states that, if the manifold is closed, there must be p plates, whose "fastening bands" do not disconnect the surface.

Addressing the problem of trying to reduce a diagram to a normal form, Heegaard notes that, in addition to isotopy of the fastening bands, a diagram may be subjected to certain moves, which, expressed in modern terminology, are: 1-handle sliding, 2-handle sliding, and eliminating a cancelling pair of handles. Thus Heegaard has intuitively arrived at the correct equivalence relation between such handle decompositions of 3-manifolds. Although "a lot of simplifications can be done by means of these moves", Heegaard nevertheless concludes that "the problem of reducing the diagram into a normal form is probably very difficult". In this of course Heegaard is also completely correct. Although the search for a "normal form" for 3-manifolds, analogous to that for surfaces, continues to be mentioned in the literature as the ultimate goal, we see that it has quickly become clear that any such normal form, if it exists, will be considerably more complicated than in the 2-dimensional case.

Heegaard next gives some simple examples of diagrams of 3-manifolds. Starting with the case of genus 1, he gives a brief discussion of the simple closed curves on a torus standardly embedded in $\mathbb{R}^3$, noting that in addition to a meridian λ (which bounds a disk "inside" the torus), and a latitude β (which bounds a disk "outside" the torus), there are also curves $[n\beta \pm \lambda]$ and $[\beta \pm n\lambda]$, defined in the obvious way. However, he states that "the complete classification is quite difficult".

Heegaard's next example is the diagram of genus p in which the fastening bands are the meridians of the string surface (the meridian–latitude terminology is extended in the obvious way to handlebodies of arbitrary genus); this manifold is the connected sum of p copies of $S^1 \times S^2$. Regarding it as the double of a handlebody, Heegaard observes that it embeds in $\mathbb{R}^4$ (by embedding the handlebody in $\mathbb{R}^3$, pushing its interior into upper half 4-space, and doubling), and also that it can be obtained by removing $2p$ disjoint 3-cells from S^3 and identifying the resulting boundaries in pairs (recall Dyck's construction [27]), in a way analogous to Klein's normal form for surfaces. Finally, Heegaard describes a genus 3 diagram of the 3-torus T^3, which he defines as the boundary of a neighborhood (the *hull*) of a torus T^2 embedded in $\mathbb{R}^4$, and notes that a similar diagram may be obtained for the hull of any surface in $\mathbb{R}^4$; this will be an orientable S^1-bundle over the surface.

Heegaard observes that associated with a given diagram, there is a second diagram, corresponding to the dual handle decomposition:

> There is a sort of dual connection between the two diagrams: the strings in one of them correspond to the plates in the other, and vice versa.

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[5] In the sense of Betti, i.e. with respect to homology "without division"; see Section 3.

Finally, he points out that a diagram expresses the manifold as the union of two solid handlebodies, with their boundaries identified in some way, and for this:

> ... it is sufficient to know [on one boundary] the system of nondisconnecting annular cuts which corresponds to the curves β on the string-surface of the other, and the system which corresponds to the curves λ.

In fact there is a certain amount of redundancy here: the manifold is actually determined by the images on the boundary of one handlebody of the meridian curves λ of the other handlebody. Thus a *Heegaard diagram* eventually came to mean two *complete systems* of curves on a closed, orientable surface F of genus p, where a complete system is a disjoint union of p simple loops whose union does not separate F; see, for example, [97].

The difficulties in using Heegaard diagrams to get "normal forms" for 3-manifolds became increasingly clear. The classification of genus 1 diagrams is relatively easy, and is done in [39], but the inherent complexity of diagrams of higher genus, even of S^3, was explicitly pointed out by Frankl [34] and Reidemeister [84]. Specifically, they gave examples of Heegaard diagrams of S^3, consisting of a complete system of curves $K_1, \dots, K_p$ on the boundary of a handlebody V of genus p (where $p = 3$ and 2, respectively), such that the manifold X obtained by adding a 2-handle to V along K_1 is not a handlebody. In Reidemeister's case ($p = 2$), X is the complement of the trefoil knot, and he points out that any knot that arises in this way will have the property that its group has a presentation with two generators and one relation. (In modern terminology, the knots in question are precisely those with *tunnel number* 1.)

In addition to the problem of analyzing different diagrams of the same underlying *Heegaard splitting*, i.e. the pair (M, F), where the *Heegaard surface* F separates M into two handlebodies, there is also the problem of analyzing different splittings of the same manifold. That this was a problem, even for S^3, was pointed out by Reidemeister in [84] (see Section 5), and Alexander gives the following discussion of these matters in his elegant paper [9] in the Proceedings of the 1932 International Congress of Mathematicians:

> One or two general remarks about the classification of manifolds according to Heegaard's program may, perhaps, be worth making. The problem divides itself naturally into two parts: (i) to determine in how many essentially different ways two canonical regions[6] of genus p can be matched together to form a manifold; (ii) to determine in how many essentially different ways a canonical region can be traced in a manifold. The first part of the problem does not seem hopelessly difficult; it is closely related to the problem of the number of essentially different one–one mappings of one surface of genus p on another. As to the second part of the problem, I have a strong suspicion that if S and S^1 are two canonical surfaces of the same genus in a manifold M then there is always a continuous deformation of the manifold M into itself carrying the surface S into the surface S^1. It would be interesting to have a proof of this hypothetical theorem even for the case where the manifold M is a hypersphere. The theorem for a general manifold M seems to be reducible to this special case.

With hindsight, this seems overly optimistic, with regard to both parts (i) and (ii). It is in fact true that all Heegaard surfaces of S^3 of a given genus are isotopic; this was proved by Waldhausen in 1968 [114]. However it is false for arbitrary 3-manifolds; the first examples, for connected sums of lens spaces, were given by Engmann [31], and, for irreducible 3-manifolds, by Birman, González-Acuña, and Montesinos [14].

[6] By a "canonical region" Alexander means a handlebody, and by a "canonical surface", a Heegaard surface.

Regarding part (i) of Alexander's comments, it does seem to be the case that it was the desire to understand 3-manifolds by means of their Heegaard diagrams that provided the initial motivation for the study of automorphisms of surfaces, by Poincaré (see Section 5), Dehn, Goeritz, and others.

A Heegaard splitting of genus p may be *stabilized* in a trivial way to give a splitting of genus $p+1$; this is the inverse of the handle cancellation observed by Heegaard. Reidemeister [84] and Singer [100] showed that any two Heegaard splittings of a given 3-manifold are stably equivalent, i.e. become isotopic after each is stabilized some number of times. This result was subsequently used by Reidemeister [85] to define certain linking invariants of 3-manifolds.

Reidemeister's proof of the stable equivalence theorem is rather sketchy, while Singer's, although quite detailed, contains a gap. The first correct published proof seems to be the one given by Craggs in [19]. For an interesting account of the proofs of Reidemeister and Singer, their difficulties, and how they can be made rigorous, see Siebenmann [99].

We have seen that Heegaard's diagrams for n-manifolds were motivated by the topological classification of 2-manifolds. Another 2-dimensional phenomenon that prompted the investigation of its higher dimensional analog was the fact that every (closed, orientable) 2-manifold is homeomorphic to a Riemann surface, that is, a branched covering of the 2-sphere. This led to the study of 3-dimensional *Riemann spaces*, in other words, branched coverings of the 3-sphere, the branch set being some link.

Heegaard discusses this in his dissertation [44, Section 13]. Assuming that the branching index around each branch curve in the manifold is 2 (or 1), he shows how to construct a diagram of the 3-manifold from the covering data. (In an earlier section, Section 8, he has done this for Riemann surfaces.) In Section 14 he gives some examples, of n-sheeted coverings M of S^3 with branch set L:

(1) $n=2$, $L=$ unknot: $M \cong S^3$.
(2) $n=2$, $L=$ 2-component unlink: $M \cong S^1 \times S^2$.
(3) $L=\nu$-component unlink: $M \cong \mathop{\#}\limits_{\nu-n+1} S^1 \times S^2$ ("a sphere with $\nu-n+1$ handles").
(4) $n=3$, $L=$ trefoil: $M \cong S^3$.
(5) $n=2$, $L=$ trefoil: $M \cong L(3,1)$.
(6) $n=2$, $L=$ Hopf link: $M \cong L(2,1) \cong \mathbb{RP}^3$.

Heegaard then applies these considerations to the subject that motivated his whole investigation, namely the study of complex algebraic surfaces. (Recall that the title of his dissertation is "Preliminary studies towards a topological theory of connectivity of algebraic surfaces".) If p is a singular point of such a surface X, and M is the 3-manifold that is the intersection of X with the 5-sphere boundary of a neighborhood of p in $\mathbb{C}^3$, then Heegaard observes that the corresponding neighborhood of p in X is homeomorphic to the cone on M. After giving some example where $M \cong S^3$, he shows that for, e.g., the curve $z^2 = x^2 - y^2$, and p the origin, the manifold M is homeomorphic to $\mathbb{RP}^3$, and so X is not a manifold near p.

Tietze, in [110, Section 18], also gives a discussion of branched coverings of S^3, explicitly mentioning Heegaard's example (4) above, and the example: $n=3$, $L=$ Hopf link: $M \cong L(3,1)$. He says:

> ... it is not known if each closed, orientable 3-manifold is homeomorphic to a "Riemann space" of this kind.

This was answered by Alexander in [3], in all dimensions: he showed that every closed, orientable n-manifold is a branched covering of the n-sphere. He concludes this short paper with the following remarks:

> In the 3-dimensional case, a Riemann space obtained by the above construction contains, in general, a network of branch lines at each of which two or more sheets coalesce. It is easy to show that, without modifying the topology of the space, the branch system may be replaced by a set of simple, nonintersecting closed curves such that only two sheets come together at a curve. The curves may, however, be knotted and linked.
>
> Three-dimensional Riemann spaces have been discussed by Heegaard and Tietze, but neither of these mathematicians seems to have been aware of their complete generality.

5. Poincaré's fifth complement

Poincaré introduces this remarkable paper [81] with the words:

> I have often had occasion to apply my thoughts to *Analysis Situs*;... I now return to this same topic, convinced that one will be able to succeed only by repeated efforts, and that the subject is important enough to merit such efforts.

He goes on to say:

> The final result that I have in view is the following. In the second complement I have shown that to characterize a manifold, it is not enough to know the Betti numbers, but that certain coefficients which I have called torsion coefficients play an important role.
>
> One may then ask if the consideration of these coefficients suffices; if a manifold all of whose Betti numbers and torsion coefficients are trivial is simply-connected in the proper sense of the word, that is to say, homeomorphic to the hypersphere.
>
> We can now answer this question....

As we have remarked above, in Section 3, Poincaré already had in hand examples of 3-manifolds with the same homology groups but different fundamental groups. However, here he proposes the more specific question of whether a homology sphere is homeomorphic to the sphere. This question had certainly occurred to Poincaré earlier; in fact his second complement concludes with the erroneous announcement that the answer is "yes" [80, p. 308].

The example, of a homology 3-sphere with nontrivial fundamental group, which answers the question, comes at the end of the fifth complement. The rest of the paper is taken up with considerations most of which are not logically necessary for the proof that this example has the desired properties, but which may be described as Poincaré's attempts to set up a theory of (his version of) Heegaard diagrams of 3-manifolds.

The most natural setting in which to express Heegaard's definitions in modern terminology is that of piecewise linear topology. By contrast, Poincaré chose to work in a smooth setting. In a remarkably far-sighted discussion, in [81, Section 2], he considers a Morse function on an m-dimensional manifold V, classifies the critical points in terms of their index, and analyzes the effect on the topology of V of passing through a critical point. (For Poincaré, V is embedded in some Euclidean space $\mathbb{R}^k$, and the Morse function corresponds to a 1-parameter family of $(k-1)$-dimensional "surfaces" $\varphi(t)$, whose intersections with V express V as the union of a 1-parameter family of $(m-1)$-dimensional submanifolds $W(t)$, possibly with singularities.)

In Section 5, specializing to dimension 3, Poincaré considers handlebodies. Specifically, he shows that if V is a 3-manifold with boundary, such that the nonsingular level surfaces $W(t)$ are connected, orientable, and increase their genus at each critical point, then there are p disjoint disks in V such that cutting V along these disks results in a 3-ball. (Thus V is a handlebody of genus p.) He thereby proves that such a manifold V is determined up to homeomorphism by p, the genus of ∂V. He also shows that, if $K_1, \ldots, K_p$ is a set of meridians for V, then $\ker(\pi_1(\partial V) \to \pi_1(V))$ is equal to the normal closure $\langle [K_1], \ldots, [K_p] \rangle$, i.e. the set of products of conjugates of $[K_1]^{\pm 1}, \ldots, [K_p]^{\pm 1}$. (We will return to this in Section 11.)

In the next section, Section 6, Poincaré considers a 3-manifold V in which $W(t)$ is a connected, orientable surface, which reduces to a point at $t = 0$ and $t = 1$, steadily increases in genus at each critical point from $t = 0$ to $t = \frac{1}{2}$, and then steadily decreases in genus from $t = \frac{1}{2}$ to $t = 1$. Then V is the union of two handlebodies V' and V'', whose common boundary is the genus p surface $W = W(\frac{1}{2})$, and on W we see meridians $K'_1, \ldots, K'_p$ for V' and $K''_1, \ldots, K''_p$ for V''. Thus we find Poincaré arriving at the concept of a Heegaard diagram by a rather different route. (Although there is no mention of it, it is hard to imagine that Poincaré was not influenced here to some extent by Heegaard's work. Certainly he was familiar with Heegaard's dissertation (recall that it was Heegaard's comments on [79] that prompted Poincaré to write his first two complements to that paper), and Heegaard had even sent him a summary of his dissertation in French [67, Section 6].)

Note, however, that Poincaré does not claim that every closed 3-manifold has a Heegaard splitting. From Poincaré's point of view, this would entail showing that one could rearrange the handles in the handle decomposition determined by the Morse function so that the 1-handles preceded the 2-handles. These considerations may be related to his false assertion in the previous section, [81, p. 90], that every closed surface in $\mathbb{R}^3$ bounds a handlebody, since (he says) it bounds a manifold "susceptible to the same [method of] generation as V" (i.e. so that there are only 1-handles). Ironically, he says that this is very surprising, as

> the various sheets of the surface might be shuffled among themselves in a complicated fashion and might form knots which it is impossible to untie without leaving 3-dimensional space.

Continuing his discussion of a closed 3-manifold V with a Heegaard splitting (V', V''), Poincaré shows that every loop in V can be homotoped into the Heegaard surface W, i.e. $\pi_1(W) \to \pi_1(V)$ is onto, and that any element in $\ker(\pi_1(W) \to \pi_1(V))$ is a product of elements in $\ker(\pi_1(W) \to \pi_1(V'))$ and $\ker(\pi_1(W) \to \pi_1(V''))$. Thus

$$\pi_1(V) \cong \pi_1(W) / \langle [K'_1], \ldots, [K'_p], [K''_1], \ldots, [K''_p] \rangle.$$

Poincaré deduces (by abelianizing) that $H_1(V)$ is the quotient of $H_1(W)$ by the subgroup generated by the homology classes of the two sets of meridians, and hence that a $2p \times 2p$ presentation matrix for $H_1(V)$ may be obtained by taking as its rows the coefficients in the expressions of the K'_i and K''_i as linear combinations of some standard basis $C_1, \ldots, C_{2p}$ for $H_1(W)$. Letting Δ denote the determinant of this matrix, Poincaré observes that if $|\Delta| > 1$ then the Betti number ("relative to homologies by division") of V is 0; if $|\Delta| = 1$ then both the Betti number and torsion coefficients vanish; and if $\Delta = 0$ then the Betti number is greater than zero.

Focusing on the case $\Delta = \pm 1$, Poincaré says that here one can ask if V is simply connected ("in the proper sense of the word", i.e. homeomorphic to S^3), and goes on:

> We shall see, and this is the principal goal of the present work, that it is not always so, and for this we will restrict ourselves to giving one example.

There follows a description of Poincaré's famous homology 3-sphere with nontrivial fundamental group. This manifold is often referred to nowadays as the Poincaré dodecahedral space, although the construction implied by this name in fact came later. Poincaré defines the manifold V in terms of a genus 2 Heegaard splitting (V', V''), with meridians K_1', K_2' and K_1'', K_2'', where K_1'' and K_2'' are explicitly drawn as unions of arcs on the 4-punctured sphere obtained by cutting the genus 2 Heegaard surface W along K_1' and K_2'. Taking a standard system of curves C_1, C_2, C_3, C_4 on W, with $C_1 = K_1'$, $C_3 = K_2'$, Poincaré writes down the elements of $\pi_1(W)$ represented by K_1'' and K_2'', in terms of C_1, C_2, C_3, C_4, and in this way obtains the following presentation for $\pi_1(V)$

$$\langle a, b\colon a^4ba^{-1}b = 1,\ b^{-2}a^{-1}ba^{-1} = 1\rangle.$$

Abelianizing gives the relations

$$3a + 2b = 0, \qquad -2a - b = 0,$$

for which $|\Delta| = 1$, showing that V is a homology sphere.

On the other hand, adjoining to the above presentation the relation $(a^{-1}b)^2 = 1$, Poincaré obtains the presentation

$$\langle a, b\colon (a^{-1}b)^2 = a^5 = b^3 = 1\rangle$$

of the icosahedral group. Since this group is nontrivial, he concludes that $\pi_1(V)$ is also nontrivial.

Finally, Poincaré says:

> There remains one question to consider:
>
> Is it possible that the fundamental group of V can be trivial, and V still not be simply connected?
>
> *In other words,*[7] is it possible to draw [on W] simple closed curves K_1'' and K_2'', so that [$\pi_1(V)$ is trivial], and that meanwhile [there do not exist pairs of meridians C_1', C_2' and C_1'', C_2'', for V' and V'', respectively, such that $|C_i' \cap C_j''| = \delta_{ij}$]?
>
> But this question would lead us too far afield.

This is the famous Poincaré conjecture. Note, however, that although the general question (is a simply-connected (in the modern sense) 3-manifold homeomorphic to S^3?) is implicit here, in fact the question that Poincaré asks is quite specific: is there a 3-manifold with a Heegaard diagram of genus 2 that is simply-connected and not homeomorphic to S^3?

(As Reidemeister points out in [84, footnote on p. 193], Poincaré's formulation of the question implicitly assumes that any genus 2 Heegaard splitting (V', V'') of S^3 has the property that there exist meridians C_1', C_2' for V' and C_1'', C_2'' for V'' such that $|C_i' \cap C_j''| = \delta_{ij}$, i.e. is equivalent to the standard genus 2 splitting. This turns out to be true, but it was not established until 1968, by Waldhausen [114].)

---

[7] My italics, C. McA. G.

It is clear that in order to arrive at his example of a nonsimply-connected homology sphere, and also in investigating his question, Poincaré must have done a good deal of experimentation with Heegaard diagrams, of genus 2 and presumably higher genus also. In particular, he must have come across many nonstandard diagrams of S^3, and realized that they did indeed represent S^3. Thus he must have been aware that such diagrams can be quite complicated. (The pitfalls here are illustrated by the discussion of Poincaré's example in the Dehn–Heegaard Enzyklopädie article [25]. There, the authors attempt to show that Poincaré's manifold is not homeomorphic to S^3 by a geometric argument, by considering the curves on a standard genus 2 Heegaard surface for S^3 that bound disks in one of the handlebodies. The proof, however, is not valid; in fact the diagram given in their paper is actually a diagram of S^3, as Dehn himself realized soon afterwards [21].)

Poincaré's detailed study of curves on surfaces, in [81, Sections 3 and 4], is also clearly motivated by Heegaard diagram considerations. In Section 3 he shows that, if F is a closed orientable surface, then an automorphism of $H_1(F)$ is induced by an automorphism of F if and only if it preserves the intersection form, and deduces that an element of $H_1(F)$ is represented by a simple loop if and only if it is indivisible. In Section 4 he gives an algorithm, using hyperbolic geometry, for deciding whether or not a loop on F is homotopic to a simple loop, and whether or not two loops are homotopic to disjoint loops.

So his remark: "But this would lead us too far afield", should probably be interpreted as indicating that his investigations of Heegaard diagrams (perhaps specifically of genus 2) were inconclusive, and that, realizing the difficulty of the problem, he decided not to pursue the matter further.

Ironically, although of course the (general) Poincaré conjecture is still open, the genus 2 case was established in 1978, with the proof of the Smith conjecture [66, p. 6].

Having decided that his question would "lead [him] too far afield", Poincaré never returned to the study of 3-dimensional manifolds. In a handful of papers, he truly created the field of topology, and 3-dimensional topology in particular. His achievements are all the more remarkable when one considers how relatively little of his time he devoted to the subject, despite being convinced of its importance. In his analysis of his own scientific works [82], for example, written in 1901, out of a total of 99 pages he devotes just over three to his work in topology; in Hadamard's 85 page account of Poincaré's mathematical work [40], topology gets two pages; and of Poincaré's over 500 publications, a mere dozen or so deal with topology, with 3-dimensional topology featuring in only two or three.

For other accounts of Poincaré's work in 3-dimensional topology see [26, 113].

6. Homology 3-spheres

Poincaré's example of a homology 3-sphere not homeomorphic to S^3 generated a good deal of interest, and the construction of other such 3-manifolds (called *Poincaré spaces* by Dehn in [22]), was for some time an identifiable theme in the literature.

The first general construction was given by Dehn in [21]. The main purpose of this short note was to point out the error in describing Poincaré's example in the Dehn–Heegaard Enzyklopädie article [25], but Dehn also took the opportunity to observe that if two copies of the complement of a knotted open solid torus in S^3 are glued together along their boundaries in such a way that a meridian of each one is identified with a latitude of the other, then the resulting manifold is a homology sphere. On the other hand, Dehn states that such a manifold cannot be homeomorphic to S^3, since it contains a torus (namely the common boundary of the two knot complements) which does not bound a solid torus on either side. Although it is indeed true that every torus in S^3 bounds a solid torus, this was not proved until later, by Alexander [4].

Another construction of Poincaré spaces was given by Dehn in his landmark 1910 paper [22]. Here he shows that, again starting with the complement of a solid toral neighborhood of a knot K in S^3, a solid torus may be attached to it in infinitely many ways (naturally indexed by the integers) so as to obtain a homology 3-sphere. Using his *Gruppenbild*, which was introduced in the same paper, he shows that for K a $(2, q)$-torus knot, the manifolds obtained by this construction all have nontrivial fundamental group (apart from the trivial attachment yielding S^3). In fact, except for a single attachment on the complement of the trefoil, the group is always infinite. In these cases the Gruppenbild is derived from a tessellation of the hyperbolic plane: the group modulo its (infinite cyclic) center is a hyperbolic triangle group. In the one exceptional case, Dehn constructs the Gruppenbild from the 1-skeleton of the dodecahedron, and shows that the group is finite, of order 120. He concludes (incorrectly, as was pointed out in [107, p. 68]) that it is isomorphic to the "icosahedral group extended by reflection". (The latter group maps onto $\mathbb{Z}_2$, while the former, being the fundamental group of a homology sphere, is perfect.) In fact the group in question is the binary icosahedral group, the inverse image of the icosahedral group under the 2-fold covering $S^3 \to SO(3)$. Curiously, Dehn makes no mention here of Poincaré's example, M_{Poin}, or the possible relation between it and his manifold M_{Dehn}.

The third member of this trio, the *spherical dodecahedral space*, M_{dodeca}, say, seems to have been first mentioned by Kneser, in a footnote to his 1929 paper [56, p. 256]. Kneser describes this manifold as the quotient of a fixed point free geometric action of the binary icosahedral group on S^3, and notes that it comes from a tiling of S^3 by 120 cells. He also states that M_{dodeca} is homeomorphic to M_{Dehn}.

In the course of their determination of all 3-dimensional spherical space forms, Seifert and Threlfall [107] also describe M_{dodeca}, and show that it can be obtained from a regular dodecahedron by identifying opposite faces by a rotation through $2\pi/10$. From this they derive a presentation of $\pi_1(M_{\text{dodeca}})$, and show that it may be transformed to both Poincaré's presentation for $\pi_1(M_{\text{Poin}})$ and Dehn's presentation for $\pi_1(M_{\text{Dehn}})$; thus all three manifolds have the same fundamental group. Seifert and Threlfall also show, again from its description as a dodecahedron with face identifications, that M_{dodeca} shares another property with M_{Poin}, namely, that it has a Heegaard splitting of genus 2.

In the text of their paper [107], Seifert and Threlfall state that they do not know whether or not any two of the manifolds M_{Poin}, M_{Dehn}, and M_{dodeca} are homeomorphic. However, in a note added in proof, they mention Kneser's reference to M_{dodeca} in [56], and say that he has shown them how to identify the complement of a certain closed curve in M_{dodeca} with the complement of the trefoil, and hence show that M_{dodeca} and M_{Dehn} are homeomorphic.

Finally, in [118], Seifert and Weber showed that all three manifolds are homeomorphic, by showing that they are all *fibered spaces* in the sense of Seifert [95], and using the results of that paper [95, Theorem 12]. We will discuss this in more detail in Section 9.

In [57], Kreines gave an example of a homology sphere obtained by identifying faces of a tetrahedron; it is also homeomorphic to M_{dodeca}.

The procedure of attaching a solid torus to the complement of a neighborhood of a knot has become known as *Dehn surgery*, and has been the focus of a lot of attention in recent years.

As well as being a source of Poincaré spaces, Dehn regarded his construction as giving a way of showing that a knot K is nontrivial: if the fundamental group of one of the homology spheres obtained by Dehn surgery on K is nontrivial, then it follows that $\pi_1(S^3 - K) \ncong \mathbb{Z}$, and so K is nontrivial. The converse, i.e. if K is nontrivial, then any manifold obtained by nontrivial Dehn surgery on K has nontrivial fundamental group, is known as the Property P conjecture, and is still unsettled.

7. Lens spaces

As we have seen, Poincaré constructed an infinite family of distinct 3-manifolds with the same Betti numbers, but in retrospect the simplest such family is the lens spaces. These were first defined by Tietze [110], as the simplest possible examples of 3-manifolds obtained by identifying faces of a polyhedron. Namely, the equator of a 3-ball is divided into p equal segments, so that the upper and lower hemispheres become p-sided polygons. These hemispherical faces are then identified by a rotation through $2\pi q/p$, where $0 \leqslant q < p$ and $(p, q) = 1$, giving a 3-manifold $L(p, q)$. If a corner is introduced along the equator of the 3-ball it assumes the lens-shaped appearance that gave these manifolds their name, the term *lens space* being introduced by Seifert and Threlfall in their paper on 3-dimensional spherical space forms [107].

Tietze notes that $L(p, q)$ may also be described as the manifold with a genus 1 Heegaard diagram consisting of a curve on the boundary of a solid torus which winds around p times latitudinally and q times meridionally ($[q\lambda + p\beta]$ in Heegaard's notation; as we have seen, the cases $L(2, 1)$ and $L(3, 1)$ were explicitly considered by Heegaard). For this reason, the lens spaces were originally referred to as *torus manifolds* [39, 56]. Tietze also observes that $L(p, q)$ is p-fold covered by $L(1, 0) \cong S^3$, and has fundamental group $\mathbb{Z}_p$, so that here we have orientable 3-manifolds with finite nontrivial fundamental group, in contrast to the situation in dimension 2.

Finally, Tietze points out that the lens spaces provide interesting examples in the context of the main problem of topology, namely the determination of necessary and sufficient conditions for two manifolds to be homeomorphic. For, Poincaré having shown that a 3-manifold is not determined by its Betti numbers and torsion coefficients, it is now natural to ask if it is determined by its fundamental group. (Earlier in his paper, Tietze had given rigorous proofs that all the then known topological invariants of a closed, orientable 3-manifold are determined by its fundamental group.) Tietze suggests that the lens spaces are potential counterexamples, and in particular raises the question of whether $L(5, 1)$ and $L(5, 2)$ (the first pair of lens spaces with the same fundamental group which are not obviously homeomorphic) are in fact homeomorphic.

In 1919 Alexander [2] showed that indeed they are not, although he seems to be unaware that the question had been raised by Tietze.

Alexander's proof is homological, and goes as follows. The lens space $L(5, 1)$ has a Heegaard diagram consisting of a solid torus A and a (5,1)-curve ℓ on ∂A, i.e. it is the union of A with another solid torus whose meridian winds around A five times latitudinally and once meridionally. Similarly, $L(5, 2)$ is defined by a solid torus A' and a (5,2)-curve ℓ' on $\partial A'$. If there were a homeomorphism from $L(5, 2)$ to $L(5, 1)$, we could assume that it takes A' into the interior of A. Then $H_1(A - A') \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by a meridian a' of

A' and a latitude b of A. If θ denotes the winding number of A' in A, then with respect to this basis

$$[\ell] = \theta a' + 5b, \quad \text{and} \quad [\ell'] = (5k+2)a' \pm 5\theta b$$

for some k, the $-$ sign allowing the possibility that the homeomorphism is orientation-reversing.

Since ℓ' bounds a disk in the complement of A', we must have

$$(5k+2)a' \pm 5\theta b = m\big(\theta a' + 5b\big), \quad \text{for some } m.$$

This readily gives $\theta^2 \equiv \pm 2 \pmod 5$, a contradiction.

More generally, Alexander's proof shows that if $L(p,q)$ and $L(p,q')$ are homeomorphic then

$$qq' \equiv \pm r^2 \pmod p, \quad \text{for some } r,$$

the sign being $+$ or $-$ according as the homeomorphism preserves or reverses orientation.

Alexander's argument was eventually formalized into the definition of the linking form $T_1(M) \times T_1(M) \to \mathbb{Q}/\mathbb{Z}$ of a 3-manifold M, where $T_1(M)$ is the torsion subgroup of $H_1(M)$. This was done in [7, 8, 85, 88] and [96]. In particular, Seifert's paper [96] gives a set of local invariants which, in the odd order case, completely classify such forms.

The condition $qq' \equiv \pm r^2 \pmod p$ did not seem to be a sufficient condition for homeomorphism, however; for example $L(7,1)$ and $L(7,2)$ appeared to be topologically distinct. The combinatorial classification of lens spaces, i.e. their classification up to *PL* homeomorphism, was finally achieved by Reidemeister in 1935 [86], using his *torsion* invariant. (This invariant was formalized and generalized to higher dimensions by Reidemeister's student Franz [35].) The result is that $L(p,q)$ and $L(p,q')$ are *PL* homeomorphic if and only if either $q \equiv \pm q' \pmod p$, or $qq' \equiv \pm 1 \pmod p$, where as usual the $\pm$ sign corresponds to the orientation character of the homeomorphism. (The sufficiency of the condition is straightforward.)

This became a classification up to homeomorphism with the proof of the Hauptvermutung by Moise in 1952 [65]. Meanwhile, Fox had outlined an approach to the topological classification, which involved considering the Alexander polynomials of knots in lens spaces, which would not require the Hauptvermutung; see [30, Problem 2]. This was implemented later by Brody [15]. (The fact which replaces the Hauptvermutung in this proof is the topological invariance of simplicial homology.)

This is a convenient place to say that although Moise's result that every 3-manifold can be triangulated, in an essentially unique way, is clearly of fundamental importance, we will not discuss it further. We remark that a simpler proof was given later by Bing [13].

The lens spaces were also natural subjects for investigations of a more algebraic topological nature. In this vein, Rueff showed [91] that there exists a degree 1 map $L(p,q) \to L(p,q')$ if and only if $qq' \equiv r^2 \pmod p$, for some r. The homotopy classification of lens spaces was obtained by Whitehead [124]: $L(p,q)$ and $L(p,q')$ are homotopy equivalent if and only if $qq' \equiv \pm r^2 \pmod p$, for some r. In particular, they are orientation-preservingly homotopy equivalent if and only if their linking forms are isomorphic. Franz [36] showed that the homotopy class of a map $L(p,q) \to L(p,q')$ is determined by the homomorphism it induces on the fundamental group, together with its degree. Together with Rueff's result, this also gives the classification up to homotopy type.

We have seen that Tietze suspected that the lens spaces provide examples of distinct 3-manifolds with isomorphic fundamental groups. He also drew attention to another apparent source of this phenomenon, at least for manifolds with boundary. In [110, pp. 96, 97] he considers the exteriors M and M' of two split links L and L' in S^3, where L consists of two copies of the right-handed trefoil K, and L' consists of a copy of K and a copy of the left-handed trefoil $-K$, the reflection of K. Thus, if X denotes the exterior of K, then M is homeomorphic to the connected sum $X \# X$, while M' is homeomorphic to $X \# -X$. Tietze notes that $\pi_1(M)$ and $\pi_1(M')$ are both isomorphic to the free product $\pi_1(X) * \pi_1(X)$. On the other hand, it appears that there is no orientation-preserving homeomorphism of S^3 taking K to $-K$, and hence no homeomorphism of S^3 taking L to L', and "hence" no homeomorphism from M to M'. The first assertion was later proved by Dehn [23], and an additional argument (which would have been available to Dehn, for example) can be given to conclude that indeed M and M' are not homeomorphic. It is interesting that these two phenomena pointed out by Tietze, namely, lens spaces, and connected summands with no orientation-reversing homeomorphism, conjecturally account completely for the failure of a closed, orientable 3-manifold to be determined by its fundamental group.

So the lens spaces provide simple examples of complex behavior in 3-manifolds: the properties of having isomorphic fundamental group, having the same homotopy type, and being homeomorphic, are all distinct. On the other hand, they are somewhat misleading; for example, it may have been that their failure to be determined by their fundamental group suggested that this was likely to be common among 3-manifolds, whereas in fact they appear to be the only irreducible examples. The lesson here seems to be: don't worry about simple counterexamples; they may be counterexamples only because they're simple.

8. Kneser's decomposition theorem

The important idea of cutting a 3-manifold along 2-spheres was introduced in the beautiful 1929 paper of Kneser [56]. Apart from his short note [55], this seems to be Kneser's only paper on 3-manifolds, but it turned out to be extremely influential.

In Section 4 of this paper, Kneser considers the operation of cutting a closed 3-manifold M along an embedded 2-sphere S, and capping off each of the resulting boundary components with a 3-ball, giving a (possibly disconnected) 3-manifold M_1. This process he calls a *reduction*. If S bounds a 3-ball in M, then M_1 is just another copy of M together with a copy of S^3, and the reduction is *trivial*. A manifold is *irreducible* if it admits only trivial reductions, i.e. if every 2-sphere in the manifold bounds a 3-ball. Kneser remarks that in order to justify the term "reduction", M_1 should be in some sense simpler than M, but that there is no reasonable topological invariant which can be used to show this. Nevertheless, he is able to prove the following finiteness theorem:

> *Associated to each 3-manifold M is an integer k with the following property: if $k + 1$ successive reductions are performed on M, then at least one of them is trivial. By means of k (or fewer) nontrivial reductions M can be transformed to an irreducible manifold.*

Before describing Kneser's proof, we discuss a result of Alexander [4], which is fundamental in this context, and which is needed in the proof. Alexander's theorem asserts that every 2-sphere in S^3 separates it into two regions, the closure of each of which is a 3-ball. In particular, S^3 is irreducible. The corresponding statement one dimension lower, that every circle in S^2 separates it into two components whose closures are disks, is the classical Schönflies theorem, and it is true with no additional hypotheses. Apparently Alexander at one time announced (but did not publish) the same result for 2-spheres in S^3 (see [5, p. 10]), but later constructed counterexamples; the first [6] was based on Antoine's necklace, and the second [5] was Alexander's famous horned sphere. Meanwhile, he gave a proof of the 3-dimensional Schönflies theorem for polyhedral 2-spheres [4]. He does this by considering the intersection of such a 2-sphere S (in $\mathbb{R}^3$) with a generic family of parallel planes. With finitely many exceptions, these will meet S transversely, each of the exceptional planes containing exactly one local minimum, local maximum, or (multiple) saddle point of S. By considering the disk bounded by an innermost simple closed curve in one of the planes containing a saddle point, Alexander replaces S by two 2-spheres, each of which is simpler than S. The theorem now follows easily by induction. (The induction starts with a sphere having only a single local minimum and a single local maximum.)

By a similar argument, Alexander also proves that any (polyhedral) torus in S^3 bounds a solid torus, a fact which was conjectured by Tietze [110]. Later, Fox [33] showed that Alexander's argument generalizes to show that any closed surface in S^3 is compressible. He used this to prove that any compact, connected 3-manifold with boundary embedded in S^3 is *homeomorphic* to the closure of the complement in S^3 of a disjoint union of handlebodies.

We now turn to Kneser's proof of his finiteness theorem.

Fix a triangulation of M, and let Σ be a disjoint union of k 2-spheres in M such that no component of $M - \Sigma$ is a punctured 3-sphere. Kneser shows that Σ may be modified so that each component of the intersection of Σ with any 2-simplex in the triangulation is an arc with its endpoints on distinct edges of the 2-simplex, and each component of the intersection of Σ with any 3-simplex is a disk.

After this, in any 2-simplex, all but at most four of the complementary regions of the intersection of Σ with that 2-simplex have a natural product structure as quadrilaterals, the possible exceptions being a triangle containing a single vertex, and a middle region meeting all three sides of the 2-simplex. Also, for each 3-simplex, if a complementary region X of the intersection of Σ with the 3-simplex meets each face of the 3-simplex in product regions, then this product structure extends over X, so that X is a *prism*.

Now the number of components of $M - \Sigma$ is at least $k - r$, where r is the first mod 2 Betti number of M. It follows from the above discussion that such a component meets every 3-simplex in prisms, unless it contains a vertex or a middle region of a 2-simplex. Therefore, letting α_i be the number of i-simplexes in the triangulation, if $k > r + \alpha_0 + \alpha_2$, then one of the components of M cut along Σ has the structure of an I-bundle over a surface, and hence is either $S^2 \times I$ or a twisted I-bundle over $\mathbb{RP}^2$ (in other words, a punctured $\mathbb{RP}^3$). In the latter case, we can collapse the corresponding 2-sphere onto the $\mathbb{RP}^2$ and repeat the argument. This shows that if $k > r + \alpha_0 + \alpha_2$ then some component of M cut along Σ is homeomorphic to $S^2 \times I$, contrary to assumption. Hence, we can take k to be $r + \alpha_0 + \alpha_2$ in the theorem.

Kneser continues:

> If you study in more detail the different possible ways of transforming by reductions a given 3-manifold into irreducible 3-manifolds, the result is the following theorem, which reduces the topological properties of all 3-manifolds to those of the irreducible ones.

Kneser then gives the following careful statement of his decomposition theorem:

> *Every 3-manifold can be expressed in the following way: take k orientable asymmetric 3-manifolds, ℓ orientable symmetric 3-manifolds, and m non-orientable 3-manifolds* ($k, \ell, m \geqslant 0$), *all irreducible, and remove a 3-ball from each; from S^3 remove $k + \ell + m + 2r + 2s$ 3-balls (where $r \geqslant 0$; $s = 0$ or 1, and $s = 0$ if $m > 0$); identify the boundary 2-spheres of the punctured manifolds with $k + \ell + m$ boundary 2-spheres of the punctured S^3; identify the remaining boundary 2-spheres in pairs, r pairs being identified in a way that is coherent with the orientation of the punctured S^3, and the last pair, if $s = 1$, so as to give a non-orientable manifold. Two 3-manifolds generated in this way are homeomorphic if and only if the numbers k, ℓ, m, r, s are the same in both cases, the 3-manifolds that are used are homeomorphic in pairs, and in the case of an orientable 3-manifold ($m = s = 0$), the orientations of the asymmetric 3-manifolds are connected in the same way in both cases.*

Kneser omits the details of the proof, but these were later elegantly supplied by Milnor [63]; one guesses that this was very much along the lines that Kneser had in mind. For the non-orientable case, see [111].

Several interesting remarks of Kneser are relegated to footnotes to his decomposition theorem. First, he defines an orientable 3-manifold to be *symmetric* if it has an orientation-reversing self-homeomorphism, and says that the simplest example of an asymmetric 3-manifold is the "torus manifold" $L(3, 1)$, or, more generally, $L(k, \ell)$, provided -1 is a quadratic nonresidue mod k. Second, he gives as examples of irreducible 3-manifolds, the 3-torus T^3 (presumably because its universal cover is $\mathbb{R}^3$), and any 3-manifold covered by S^3. It is here that he mentions in passing that an example of a manifold of this second type is the homology sphere with nontrivial finite fundamental group constructed by Dehn from the trefoil knot. Another footnote makes a reasoned plea for the use of the term "path group" instead of "fundamental group", a plea that seems to have gone unheeded.

Going back to Kneser's proof of his finiteness theorem, this beautiful argument had far-reaching consequences in the work of Haken about thirty years later. Haken observed that Kneser's argument can be applied to a system of disjoint, incompressible (closed) surfaces in a (compact) irreducible 3-manifold M, to show that there is an integer $k(M)$ with the property that the number of such surfaces, no two of which cobound a product in M, is at most $k(M)$. This finiteness theorem allows him to prove that every irreducible manifold which contains an incompressible surface (these are now called *Haken manifolds*), has a *hierarchy*, in other words, it can be reduced to a disjoint union of 3-balls by successively cutting it along incompressible surfaces. This was used to great effect by Waldhausen, to prove, for example, that two Haken manifolds with isomorphic fundamental groups are homeomorphic [115], that the universal cover of a Haken manifold is $\mathbb{R}^3$ [115], and that the fundamental group of a Haken manifold has solvable word problem [116].

Again based on Kneser's idea of controlling his surfaces (spheres) by making them have nice intersections with the simplices of a fixed triangulation of the manifold, Haken developed an algorithmic theory of such *normal surfaces* [42]. This ultimately led, with the work of Waldhausen, Johannson, Jaco-Shalen, and Hemion's solution of the conjugacy problem for automorphisms of surfaces, to the solution of the homeomorphism problem for Haken manifolds; see [117].

So the idea, in Heegaard's words, of "cutting a manifold until it is simply connected", is realized in Haken's concept of a hierarchy, and leads to a solution of the homeomorphism problem for a large class of 3-manifolds, although the notion of a "normal form" survives only as a nebulous logical construct.

Finally, we mention that, very recently, Rubinstein [90] (see also Thompson [106]) has solved the homeomorphism problem for S^3, also using normal surfaces, but in a very different way. So we see how important Kneser's few pages have been for the theory of 3-manifolds.

9. Geometric 3-manifolds

We have already seen how early approaches to the study of 3-manifolds naturally took the form of pursuing analogies with the theory of 2-manifolds. Another feature of 2-manifolds which was well known from the work of Klein and others was that every closed surface can be given a spherical, Euclidean, or hyperbolic structure. That is, it can be represented as the quotient of either the 2-sphere S^2, the Euclidean plane E^2, or the hyperbolic plane H^2, by a group of isometries acting freely and properly discontinuously. This group is of course isomorphic to the fundamental group of the 2-manifold.

The fact that the lens space $L(p,q)$ is the quotient of such an action on S^3 by a cyclic group of order p is essentially in Tietze [110], and was made explicit in Hopf [47], who also gave other examples of spherical 3-manifolds, with noncyclic fundamental groups. In addition, Hopf proved that any n-manifold with a complete Riemannian metric of constant curvature is a quotient of S^n, E^n, or H^n by a free properly discontinuous action of a group of isometries. By finding all the finite subgroups of $SO(4)$ that act freely on S^3, Seifert and Threlfall, in [107, 108], gave a complete description of all spherical 3-manifolds.

Their classification can be roughly described as follows. The quotient of $SO(4)$ (the group of orientation-preserving isometries of S^3) by its center $\{\pm id\}$ is isomorphic to $SO(3) \times SO(3)$, so a finite subgroup G of $SO(4)$ gives rise to two finite subgroups G_L and G_R of $SO(3)$. Now the finite subgroups of $SO(3)$ are the finite cyclic groups C_n, the dihedral groups D_{2n} of order $2n$, and the tetrahedral, octahedral, and icosahedral groups T, O, and I, of orders 12, 24, and 60. If G acts freely on S^3, then G_L (say) must be cyclic, and G can then be described as being of cyclic, dihedral, tetrahedral, octahedral, or icosahedral type, according to the type of G_R. The groups of cyclic type are cyclic, and the corresponding 3-manifolds are the lens spaces. The groups of dihedral, tetrahedral, octahedral, and iccosahedral type include the corresponding binary groups D^*_{2n}, T^*, O^*, and I^*, while the dihedral and tetrahedral types include additional families $D'_{m,n}$ and T'_m. The general group of a given type is the direct product of any one of these with a cyclic group of relatively prime order.

The 3-manifolds M with fundamental groups G of dihedral type are the *prism spaces*: if $G_L \cong C_m$ and $G_R \cong D_{2n}$, then M can be obtained by suitably identifying the faces of a $2mn$-sided prism. A special case of this is quaternionic space, $m = 1, n = 2$, obtained by identifying opposite faces of a cube, as described by Poincaré (see Section 3). The spherical dodecahedral space M_{dodeca} is of icosahedral type, with $G_L = 1$ and fundamental group G

the binary icosahedral group I^*. It is the only homology 3-sphere (apart from S^3) among the spherical 3-manifolds.

The lens spaces and prism spaces also appear, in a different context, in [93]. There, Seifert classifies the 3-manifolds that can be obtained from a solid torus by identifying its boundary with itself via some involution. These manifolds fall into three classes, according to the nature of the involution: the first are the lens spaces and $S^1 \times S^2$, the second are the prism spaces (including the lens spaces $L(4q, 2q-1)$, and $\mathbb{RP}^3 \# \mathbb{RP}^3$, as degenerate cases), and the third consist of $S^1 \times \mathbb{RP}^2$ and the twisted S^1-bundle over S^2.

The question remains whether every 3-manifold with finite fundamental group G is spherical. A more modest goal would be to show that at least G is isomorphic to $\pi_1(M)$ for some spherical 3-manifold M. These questions are still open, but progress was made in 1957 by Milnor, who proved [62] that any such G has at most one element of order 2. Combining this with the fact that the cohomology of G must have period 4, he deduced that any counterexample G to the second assertion must belong to one of two infinite families $Q(8n, k, \ell)$ and $O(48r)$. The second family, and half the first family, were subsequently ruled out by Lee [58].

An important offshoot of the work of Seifert and Threlfall came from their observation that any finite subgroup of $SO(4)$ which acts freely on S^3 commutes with an S^1 subgroup of $SO(4)$, and hence this S^1-action descends to the quotient manifold M, giving M a (singular) fibering by the orbits of the action. This motivated Seifert to investigate 3-manifolds which can be fibered by circles in this fashion, now called *Seifert fibered spaces*. (The special case of circle tangent bundles of surfaces had been studied earlier by Hotelling [50, 51], as the 3-manifolds of states of motion of dynamical systems.) In the introduction to his work [95] on fibered spaces (see also the translation by W. Heil in [97]) Seifert says:

> The question that underlies this paper is the homeomorphism problem for 3-dimensional closed manifolds. The fundamental theorem of surface topology tells us how many topologically distinct 2-manifolds there are. The methods used to prove this have not yet been generalized to three or more dimensions. There are two ways to approach the 3-dimensional problem. The first is to examine the regions of discontinuity[8] of 3-dimensional metric groups of motions. Whereas in two dimensions every closed surface appears as the region of discontinuity of a fixed point free group of motions, there are 3-manifolds for which this does not hold. The regions of discontinuity of 3-dimensional spherical actions are endowed with a certain fibration; the fibers are the orbits of a continuous group of motions of the sphere. ... This leads us to the second approach: instead of investigating a complete system of topological invariants of 3-dimensional manifolds, we search for a system of invariants for fiber-preserving maps of fibered 3-manifolds. This problem is completely solved in this paper. Of course these invariants refer to the fibering of the manifold, not to the manifold itself, so that the question remains open, whether two spaces with different fibrations are topologically distinct. Moreover, there are 3-manifolds that do not admit any fibration. Nevertheless, in many cases the fiber invariants can be used to decide whether 3-manifolds are homeomorphic.

Seifert's paper is a masterpiece of content and clarity. He builds up from scratch the complete and rich theory of his fibered spaces, and in fact his account left little to be added for several decades. With the torus decomposition theorem of Johannson, and Jaco and Shalen, the Seifert fibered spaces emerged as one of the basic building blocks of Haken manifolds,

[8] I.e. quotients.

and this role has been further clarified and emphasized by the work of Thurston on geometrization of 3-manifolds.

Seifert defines a 3-dimensional *fibered space* to be a closed 3-manifold M which is a disjoint union of circles (*fibers*), such that each fiber has a solid torus neighborhood, consisting of fibers, which are the core of the solid torus together with curves that wind around the core α ($\geqslant 1$) times latitudinally and ν times meridionally. If $\alpha > 1$ then the core is a *singular* fiber of M, of *multiplicity* α. The *base* of the Seifert fibration is the quotient surface obtained from M by identifying each fiber to a point.

Seifert shows that oriented fibered spaces, up to orientation- and fiber-preserving homeomorphism, are classified by $(F; b; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$, where F is the topological type of the base surface, (α_i, β_i) are the suitably normalized invariants of the singular fibers, $1 \leqslant i \leqslant r$, and b is the Euler number of a certain associated circle bundle over F, i.e. a fibered space with no singular fibers. He also gives a similar, but more involved, classification of the non-orientable fibered spaces.

In [95, 108], it is shown that the Seifert fibered spaces M with $\pi_1(M)$ finite are precisely the spherical 3-manifolds. These are S^3 (whose Seifert fibrations correspond to pairs of nonzero coprime integers m, n, the nonsingular fibers being (m, n)-torus knots); the lens spaces (which have Seifert fibrations with base S^2 and one or two singular fibers); and the Seifert fiber spaces with base S^2 and three singular fibers, whose multiplicities form one of the Platonic triples $(2, 2, n)$, $n \geqslant 2$, $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$, corresponding to the groups of dihedral, tetrahedral, octahedral, and icosahedral type, respectively. Some of the prism spaces also have Seifert fibrations with base $\mathbb{RP}^2$ and one singular fiber.

Turning to Poincaré spaces, i.e. homology 3-spheres not homeomorphic to S^3, Seifert shows that for any sequence $\alpha_1, \ldots, \alpha_r$ of $r \geqslant 3$ pairwise coprime integers $\geqslant 2$, there exists a Seifert fibered Poincaré space, with base S^2 and r singular fibers of multiplicities $\alpha_1, \ldots, \alpha_r$. Conversely, every Seifert fibered Poincaré space is of this form, and two such are homeomorphic if and only if the corresponding sequences of multiplicities $\alpha_1, \ldots, \alpha_r$ are the same, up to order, in which case they are fiber-preservingly homeomorphic. The only Seifert fibered Poincaré space with finite fundamental group is the spherical dodecahedral space, with three singular fibers of multiplicities 2, 3, and 5.

Recalling Dehn's construction of Poincaré spaces by surgery on knots, Seifert shows that the Poincaré spaces obtained in this way from an (m, n)-torus knot are precisely those that have Seifert fibrations with three singular fibers of multiplicities m, n, and $|qmn - 1|$, for some $q \neq 0$. In particular, the manifolds M_{Dehn} and M_{dodeca} are homeomorphic.

Seifert also discusses branched coverings of Seifert fibered spaces, and shows that the Seifert fibered Poincaré spaces can be realized in several different ways as branched coverings of S^3, with branch set a collection of nonsingular fibers in some Seifert fibration of S^3. In particular, if $\alpha_1, \alpha_2, \alpha_3$ are pairwise coprime integers $\geqslant 2$, then the α_i-fold cyclic branched covering of the (α_j, α_k)-torus knot is the Seifert fibered Poincaré space with three singular fibers of multiplicities $\alpha_1, \alpha_2, \alpha_3$, where $\{i, j, k\} = \{1, 2, 3\}$.

This brings us to the proof in [118] that M_{Poin} is homeomorphic to M_{dodeca}. Seifert and Weber start with the genus 2 Heegaard diagram of $M = M_{\text{Poin}}$ given by Poincaré in [81], expressing M as the union of two genus 2 handlebodies V and V', with meridians K_1, K_2 and K_1', K_2', respectively. They note that there is an orientation-reversing involution of the Heegaard surface, with fixed point set a simple closed curve C, which interchanges K_i and K_i', $i = 1, 2$. (Interestingly, this was also observed by Poincaré [81, p. 108], although he made no use of it.) Hence there is an involution on M, interchanging V and V', with fixed point set C. Seifert and Weber show that the quotient of the pair (M, C) by this involution is (S^3, K), where K is a (3,5)-torus knot. Thus M is the 2-fold covering of S^3 branched along K, and hence is the unique Seifert fibered homology sphere with singular fibers of multiplicities 2, 3, and 5.

Turning to Euclidean 3-manifolds, these were classified by Nowacki [70], and Hantzsche and Wendt [43], independently. There are precisely 10 of them, 6 orientable and 4 non-orientable. They are all covered by the 3-torus T^3. Nowacki also classified the open Euclidean 3-manifolds; here there are 4 orientable and 4 non-orientable examples. Nowacki's proof is based on the classification of the 3-dimensional crystallographic groups, while that of Hantzsche and Wendt is more direct.

What about the hyperbolic case, which in dimension 2 is the generic one? Hantzsche and Wendt conclude their paper by saying:

> The Euclidean space form problem is hereby completely settled, and the spherical case has been done in [107, 108]. Much harder is the question of hyperbolic space forms, of which one knows only a few examples.

The first example to be given, of a discrete subgroup of $PSL_2(\mathbb{C})$ (the group of orientation-preserving isometries of H^3) with a fundamental domain of finite volume, was the group $PSL_2(\mathbb{Z}[i])$, described by Picard [76]. More generally, Bianchi studied the groups $PSL_2(R)$, where R is the ring of algebraic integers in an imaginary quadratic number field [12]. However, no-one at that time seems to have found, or looked for, torsion-free subgroups of finite index of these groups, which would give rise to (cusped) hyperbolic 3-manifolds of finite volume. The first hyperbolic 3-manifold of finite volume was described in the 1912 thesis of Gieseking [38], a student of Dehn. (See also [60, Chapter V].) This manifold can be obtained from a regular ideal tetrahedron in H^3 by suitably identifying its faces in pairs. It is non-orientable, and turns out to be the unique noncompact hyperbolic 3-manifold of minimal volume [1].

The first examples of closed hyperbolic 3-manifolds were constructed by Löbell in 1931 [59]; (in the preface to his paper he thanks Koebe for "expressing, in conversation, the desire that the question of the existence of such examples should be decided"). Löbell starts by constructing a 3-dimensional hyperbolic polyhedron whose faces are two right angled hexagons and 12 right angled pentagons. By suitably assembling copies of this polyhedron he builds a compact hyperbolic 3-manifold, whose boundary is totally geodesic and consists of four isometric copies of a surface of genus 2. Taking a finite number of copies of this manifold, and identifying the boundary components in pairs, he then obtains infinitely many closed hyperbolic 3-manifolds, which can be chosen to be either orientable or non-orientable.

A more symmetrical example was described by Seifert and Weber in [118]. This manifold, the *hyperbolic dodecahedral space*, or *Seifert–Weber manifold*, comes from a tiling of H^3 by regular dodecahedra with dihedral angles $2\pi/5$; it is the quotient of H^3 by a fixed point free group of isometries having one of these dodecahedra as a fundamental domain. It can be obtained from a single copy of the dodecahedron by identifying opposite faces by a rotation through $3\pi/5$. They also show that it is a 5-fold cyclic branched covering of S^3, with branch set the Whitehead link.

Permitting ourselves to look ahead, some more examples of closed hyperbolic 3-manifolds were constructed by Best in 1971 [11], using other regular hyperbolic 3-dimensional polyhedra. In 1975 Riley [89] showed that the complement $S^3 - K$ of the figure eight knot K has a complete hyperbolic structure, by finding an explicit discrete, faithful representation of $\pi_1(S^3 - K)$ in the Bianchi group $PSL_2(\mathbb{Z}[e^{2\pi i/3}])$. Nevertheless, at that time it was still the case that only a few examples of hyperbolic 3-manifolds were known. The situation changed dramatically with the work of Thurston, however (see [109]), who showed that hyperbolic 3-manifolds are plentiful, and indeed presented much evidence for his *geometrization conjecture*, which would imply, for example, that any closed orientable 3-manifold which satisfies certain obvious necessary conditions for it to be hyperbolic, namely that it is irreducible, and its fundamental group has no free Abelian subgroup of rank 2, is in fact hyperbolic. So it appears that, just as in dimension 2, hyperbolic geometry is "generic" in dimension 3.

10. The state of play up to 1935

The year 1935 is a convenient place at which to pause and take stock of the state of 3-dimensional topology. By this time the foundations of what we would now call geometric topology had become sufficiently well established that a textbook could be written, Seifert and Threlfall's famous "Lehrbuch der Topologie", published in 1934. (An English translation appeared in 1980 [97].) One chapter of that book is specifically devoted to 3-manifolds.

Let us summarize what has been achieved. The fundamental group has emerged as an important invariant, although it is known that there are nonhomeomorphic 3-manifolds (lens spaces) with isomorphic groups. The homology of a 3-manifold is determined by its fundamental group, and is now seen as a very weak invariant; in particular there are infinitely many homology 3-spheres. There is a complete description of all 3-dimensional spherical manifolds, and of the handful of Euclidean ones. There are also some examples of hyperbolic 3-manifolds. Seifert has given a complete description of all his fibered spaces, and classified them up to fiber-preserving homeomorphism. Dehn has shown how to construct 3-manifolds by "surgery" on knots.

Three methods are known by which all 3-manifolds may be constructed: Heegaard diagrams, identification of faces of polyhedra, and branched coverings of the 3-sphere. However, none of these methods has led to anything approaching a classification. The situation is summarized well by Seifert and Threlfall [97, p. 228]:

> The construction of 3-dimensional manifolds has been reduced to a 2-dimensional problem by means of the Heegaard diagram. This problem is the enumeration of all Heegaard diagrams. Even if the diagrams could all be enumerated, the homeomorphism problem in 3 dimensions would not be solved because a criterion is still lacking for deciding when two different Heegaard diagrams generate the same manifold. The enumeration has been carried out successfully in the simplest case, that of Heegaard diagrams of genus 1, but the problem of coincidence of manifolds, that is, the homeomorphism problem for lens spaces, has not been solved even here.
>
> Another way to attempt the enumeration of all 3-dimensional manifolds would be to construct all polyhedra having pairwise association of faces. This also is a 2-dimensional problem and it has met with as little success at solution as the problem of enumerating the Heegaard diagrams.
>
> It is known from the theory of functions of complex variables that one can obtain any closed orientable surface as a branched covering surface of the 2-sphere, where the branching occurs at finitely many points. Corresponding to this result, it is possible

> to describe each closed orientable 3-dimensional manifold as a branched covering of the 3-sphere. In this case the branching occurs along closed curves (knots) which lie in the 3-sphere. Here also the enumeration and distinguishing of individual covering spaces leads to unanswered questions. On occasion the same manifold can be derived as branched coverings of the 3-sphere with quite distinct knots as branch sets; as an example, three different branch sets are known for the spherical dodecahedron space.

The only general result on the structure of 3-manifolds is Kneser's existence and uniqueness of prime decompositions.

In the introduction to their chapter on n-dimensional manifolds [97, p. 235], Seifert and Threlfall say:

> Because of their clear geometric significance, homogeneous complexes play a distinctive role among the complexes. We have given the name "manifolds" to the homogeneous complexes in 2 and 3 dimensions and we have attempted to gain a complete view of their properties. Our attempt was successful in 2 dimensions. In 3 dimensions we did not get further than a presentation of more or less systematically arranged examples. The complete classification of n-dimensional manifolds is a hopeless task at the present time.

11. Dehn's lemma and the loop theorem

Dehn's 1910 article, "On the topology of 3-dimensional space" [22], contains the following statement, which he refers to simply as *the lemma*, "because of its important place" in the paper.

DEHN'S LEMMA (1). *Let X be a 2-complex in the interior of an n-dimensional manifold M, $n > 2$. On X, let the curve C bound a singular disk D. If D has no singularities on its boundary, then C bounds an embedded disk in M.*

We will discuss the context of this rather curious statement later.

As Dehn says, the lemma is clearly true if $n > 3$; the interesting case is when $n = 3$. In this case, a little thought shows that it may be restated as follows:

DEHN'S LEMMA (2). *Let C be a simple loop on the boundary of a 3-manifold M, which bounds a singular disk in M. Then C bounds an embedded disk in M.*

Note that the hypothesis is equivalent to the statement that C is null-homotopic in M. Perhaps the most natural statement of this kind, which dispenses with the assumption that C is simple, and asserts that if a 3-manifold contains a nontrivial singular disk (homotopical information) then it contains a nontrivial embedded disk (topological information), is the following, which might be called the

DISK THEOREM. *Let M be a 3-manifold and let F be a boundary component of M such that $\pi_1(F) \to \pi_1(M)$ is not injective. Then M contains an embedded disk D, with ∂D contained in F, such that $[\partial D] \neq 1 \in \pi_1(F)$.*

By taking F to be an open annular neighborhood of C, we see that the disk theorem implies Dehn's lemma. On the other, it is implied by Dehn's lemma together with the

LOOP THEOREM. *Let M be a 3-manifold and let F be a boundary component of M such that $\pi_1(F) \to \pi_1(M)$ is not injective. Then there is an essential simple loop in F which is null-homotopic in M.*

We will return to these statements later. But first we note that arguments with singular disks in connection with the fundamental group of a 3-manifold appear in Poincaré's work. For example, in [81], he wishes to show that if $C_1, \ldots, C_p$ are (homologically independent) disjoint simple loops on the boundary of a handlebody V of genus p, which are null-homotopic in V, then they bound disjoint embedded disks in V. The "innermost disk" cutting and pasting argument that he uses to make the disks disjoint is valid if they are already embedded; however, on the latter point he merely says:

> ... an analogous argument will show that since the curves are embedded one may always suppose that the disks [that they bound] are surfaces without double curves.

This is precisely Dehn's lemma.

This fact about embedded disks in a handlebody is not needed for the discussion of Poincaré's homology sphere, but, as we have mentioned in Section 5, in order to compute the fundamental group, the following fact is: if $C_1, \ldots, C_p$ is a system of meridians on the boundary of a handlebody V of genus p, then any element in the kernel of $\pi_1(\partial V) \to \pi_1(V)$ is a product of conjugates of $[C_1]^{\pm 1}, \ldots, [C_p]^{\pm 1}$. Poincaré proves this by taking a disk D in V bounded by a loop C on ∂V, and considering the arcs of intersection of D with disks $A_1, \ldots, A_p$ bounded by $C_1, \ldots, C_p$. The disks A_i are certainly disjoint and embedded, but Poincaré appears to assume that C and D are also nonsingular; however, if one interprets his argument as applying to the inverse image of the union of the A_is under a map of a disk into V, then it is in fact correct. A similar remark applies to his proof that given a Heegaard splitting (V', V'') of a 3-manifold, a loop on the Heegaard surface W is null-homotopic in the manifold if and only if it is a product of elements in the two kernels $\ker(\pi_1(W) \to \pi_1(V'))$ and $\ker(\pi_1(W) \to \pi_1(V''))$.

Dehn attempted to prove his lemma by using the cutting and pasting procedure (he calls the operation a *switch*, or *Umschaltung*), that is hinted at in Poincaré, to remove the singularities of the given singular disk, but although he did give a detailed argument, it was, as is well known, faulty. The error was pointed out by Kneser in a footnote added in proof to his paper [56], and privately in a letter to Dehn dated 22 April, 1929 (see [24, p. 87]). Kneser himself was trying to prove the following:

KNESER'S HILFSATZ. *Let F be a closed surface in a 3-manifold M such that $\pi_1(F) \to \pi_1(M)$ is not injective. Then there exists a disk D in M such that $D \cap F = \partial D$ is an essential loop in F, and the only singularities of D are on ∂D.*

Note that by allowing singularities on ∂D Kneser is taking into account the possibility that F is 1-sided in M.

Kneser starts with a singular disk D in M whose boundary is an essential loop in F, and first shows, by considering the intersection of D with F, that one may assume that $D \cap F = \partial D$. (Modulo this simplification, the Hilfsatz is equivalent to the disk theorem, by cutting M along F.) There follows a cutting and pasting argument to eliminate the double *arcs* of D, and finally an appeal to Dehn's lemma to get a disk of the desired type.

The model for the cutting and pasting approach to Dehn's lemma and Kneser's Hilfsatz is the case where the singular disk has only double points (i.e. no triple points), in which case a proof by these methods may readily be obtained – indeed as we have seen, this is essentially contained in Poincaré's work. More generally, a switch can be used to eliminate any *simple* double curve, i.e. one that comes from the identification of two disjoint simple closed curves in the nonsingular preimage of the disk. But Dehn's attempt to carry this through when the double curves themselves have singularities is unsuccessful; it is not clear in general that the cutting and pasting can be done consistently in the presence of these triple points. The sort of difficulty that arises is illustrated by an example given by Johannson in [52], which he states that Dehn was (by that time) also aware of. Kneser's argument for removing double arcs (which he gives in the case where there are no triple points, the general case being said to follow analogously), is also subject to the same criticism.

Johannson [52] gave necessary conditions for the realizability of *Dehn diagrams* (i.e. patterns of immersed circles) as the double curves of a singular disk in a 3-manifold, and in particular showed that the example mentioned above could not in fact be realized. He states that in fact in all known examples the cutting and pasting argument can be successfully carried out, and hence

> one can still hope that it might be possible to prove the lemma by suitably selected switches.

In [53] Johannson showed that if Dehn's lemma is true for orientable 3-manifolds then it is also true for non-orientable ones.

There the situation remained until the ground-breaking work of Papakyriakopoulos in 1957.

Before discussing this, let us briefly return to the papers of Dehn and Kneser, to see the uses to which they put their "lemmas".

Starting with Kneser, as an application of his Hilfsatz, he proves that every closed surface F in S^3 can be obtained from a 2-sphere by adding handles. In particular, this recovers (or would recover, if the proof of the Hilfsatz were correct) Alexander's theorem that every torus in S^3 bounds a solid torus [4]. The argument is straightforward. If F is not a 2-sphere, then the map $\pi_1(F) \to \pi_1(S^3) = 1$ is not injective, and hence, by the Hilfsatz, there is an embedded disk D in S^3 such that $D \cap F = \partial D$ is essential in F. Compressing F along D gives a simpler surface F', from which F is obtained by adding a handle. The result now follows by induction. Here we first find explicitly the important idea of compressing a surface F in a 3-manifold M, using the noninjectivity of $\pi_1(F) \to \pi_1(M)$, which Kneser used in his "proof" of his "conjecture" (see Section 12), and which played a central role in the later work of Stallings and Waldhausen.

Turning to Dehn's paper, the last chapter consists of two sections. In the first he shows that every closed 3-manifold M can be obtained by sewing a 3-ball along its boundary onto a *seam surface* N_2 in M. (This 2-complex N_2 is exactly Heegaard's "nucleus".) Letting N_1 be the 1-skeleton of N_2, Dehn notes that a neighborhood of N_1 in M is a (possibly non-orientable) solid handlebody, whose complement is also a solid handlebody. He concludes that every closed 3-manifold has a Heegaard splitting, although, oddly, he does not mention Heegaard here at all. He also notes that this implies that every closed 3-manifold is the union of four 3-balls.

The second section is described by Dehn in the introduction to his paper (see [24]):

> Section 2 deals with the important problem of the topological characterization of ordinary space, without, however, resolving the problem. It treats the question of how

> ordinary space may be topologically defined through the properties of its closed curves, and how to make it possible to decide whether or not a given space is homeomorphic to ordinary space. The history of this problem began when first Heegaard (Diss. Copenhagen 1898) and then Poincaré (Pal. Rend. v.13 and Lond. M.S. v.32) pointed out that in order to characterize ordinary space it does not suffice to assume that each curve bounds, possibly when multiply traversed. Indeed the manifolds with *torsion* show this. Then Poincaré proved in Pal. Rend. 1904, by construction of a "Poincaré space" that it is even insufficient for each curve to bound when traversed once.
>
> It now is natural to investigate whether it suffices to suppose that each curve in the space bounds a disk. This is also suggested at the end of Poincaré's work. However, the reduction of the problem given in the present work does not appear to lead directly to a solution. A deeper investigation of the fundamental groups of two-sided closed surfaces seems to be unavoidable.

Dehn's "reduction" of the Poincaré conjecture is the following. He notes that one may assume that N_1 consists of a wedge of circles. Since the manifold M is simply connected by hypothesis, each of these circles bounds a singular disk. *If* it were possible to choose these disks to have no singularities on their boundaries, then Dehn's lemma would give a system of embedded disks, with the same boundaries, meeting only at a single point. A neighborhood of the union of these disks would be a 3-ball, whose boundary S is contained in the solid handlebody A that is the complement of a neighborhood of N_1. Dehn now asserts that the 2-sphere S bounds a 3-ball in A, implying that M is the union of two 3-balls along their boundaries, and therefore homeomorphic to S^3. (Of course, the fact that a 2-sphere in a handlebody bounds a 3-ball requires proof and was not available at the time; if the handlebody is orientable it follows from Alexander's theorem that a (tame) 2-sphere in S^3 separates it into two 3-balls [4], and can be proved for non-orientable handlebodies by passing to the orientable 2-fold cover.)

Although Dehn's lemma finds a very important and natural application in the theorem that a knot with group $\mathbb{Z}$ is trivial, both the title of Dehn's paper and the mention of a 2-complex in the statement of the lemma (presumably Dehn had in mind the seam surface N_2) suggest that Dehn's real motivation was the Poincaré conjecture. Incidentally, Dehn's approach formed the basis for later attacks on this problem, notably by Haken.

As we have noted above, Dehn's lemma is something of a hybrid, and it alone does not enable one to translate purely homotopy theoretic information into topological information. For this reason, Papakyriakopoulos formulated the loop theorem, which he proved in [71]. He explains [74] that his motivation was the following characterization of handlebodies.

CONJECTURE H. *If M is a compact 3-manifold with boundary an orientable surface of genus g, and $\pi_1(M, \partial M) = 1$, then M is a handlebody of genus g.*

This is turn was apparently motivated by the Poincaré conjecture:

> Some years ago I was working on the Poincaré conjecture, and I tried to prove it by proving [Conjecture H]. But I failed, and I may say that I am now convinced that this is not the way to attack the Poincaré conjecture. However, the loop theorem, Dehn's lemma, Poincaré conjecture, and some results from algebraic topology imply [Conjecture H], see [72, Theorem (19.1), p. 297]. This was the reason I worked on the loop theorem, whose proof led me to the proof of Dehn's lemma and the sphere theorem.

The key idea that enabled Papakyriakopoulos to prove the loop theorem, Dehn's lemma, and the sphere theorem, was the use of covering spaces. Another, more elementary, principle that is used in all three proofs is the relation between the first homology of a 3-manifold and that of its boundary. We have already seen a special case of this for submanifolds of $\mathbb{R}^3$ in Poincaré's work; (see Section 3). Of more direct relevance here is the fact, proved by Kneser in [55], that if M is a 3-manifold such that $H_1(M; \mathbb{Z}_2) = 0$, then any two 1-cycles in ∂M have even intersection number. It follows that each component of ∂M is *planar*, i.e. embeds in the 2-sphere. Later, Seifert proved [94] that if M is compact and orientable, with boundary components of genera $p_1, \ldots, p_r$, then $\beta_1(M) \geqslant p_1 + \cdots + p_r$. In particular, if ∂M does not consist of 2-spheres, then $H_1(M)$ is infinite.

Here is a summary of Papakyriakopoulos' proof of the loop theorem. Let C be a loop in a boundary component F of a 3-manifold M, which is essential in F but null-homotopic in M. Let $p : \widetilde{M} \to M$ be the universal cover. Since C is null-homotopic in M, it lifts to a loop $\widetilde{C}$, say, in $p^{-1}(F)$. Now the crucial observation is that the singularities of C are the images under p of, firstly, the singularities of $\widetilde{C}$, and secondly, the intersections of $\widetilde{C}$ with its translates $\tau(\widetilde{C})$ under nontrivial elements τ of the group of covering transformations of $\widetilde{M}$. Hence, one wants to replace $\widetilde{C}$ by an essential loop $\widetilde{C}^*$ in $p^{-1}(F)$ such that (i) $\widetilde{C}^*$ is simple, and (ii) $\widetilde{C}^* \cap \tau(\widetilde{C}^*) = \emptyset$ for all τ. Note that $\widetilde{C}^*$ is automatically null-homotopic in $\widetilde{M}$ since $\pi_1(\widetilde{M}) = 1$. Then $C^* = p(\widetilde{C}^*)$ will be a simple essential loop in F which is null-homotopic in M. Condition (i) is easy to satisfy, and Papakyriakopoulos shows, by a delicate combinatorial argument, that (ii) can also be achieved. The important fact here is that $p^{-1}(F)$ is planar, by Kneser's result.

Whitehead had earlier proved a special case of the loop theorem, by a direct cutting and pasting argument: if C is a simple loop in the boundary of a 3-manifold M such that C^n is null-homotopic in M for some $n > 0$, then C is null-homotopic in M [122].

As Papakyriakopoulos says in [74]:

> Having observed ... that the loop theorem and Dehn's lemma are problems of the same kind, and having proved the loop theorem, the question arises naturally: *can we use the same method, or at least a modification of it, to prove Dehn's lemma*? The answer is affirmative ....

To prove Dehn's lemma, Papakyriakopoulos came up with his famous *tower construction*. In this, a tower of coverings is constructed, by starting with a neighborhood V_0 of the given singular disk $D_0 \subset M = M_0$, taking the universal covering M_1 of V_0, and lifting the map of the disk into V_0 to get a singular disk $D_1 \subset M_1$; now taking a neighborhood V_1 of D_1, taking the universal covering M_2 of V_1, and so on. Since D_i has fewer singularities than D_{i-1}, the tower must terminate, at $D_n \subset V_n \subset M_n$, say. In particular, since $\pi_1(V_n) = 1$, ∂V_n consists of 2-spheres, by Kneser's result.

Recall that we may assume that D_0 has no simple double curve, for otherwise D_0 could be simplified by a Dehn switch. Papakyriakopoulos distinguishes two cases at the top of the tower: (1) D_n is singular, and (2) D_n is nonsingular.

In case (1), since ∂V_n consists of 2-spheres, ∂D_n bounds a disk in ∂V_n. Papakyriakopoulos shows that this disk, when projected down the tower, gives a disk D_0^* in M, with $\partial D_0^* = C$, which (using the fact that D_0 has no simple double curves) has fewer triple points than D_0.

In case (2), first Papakyriakopoulos notes that ∂V_{n-1} does not consist of 2-spheres. (If $n = 1$, this is because otherwise the triple points of D_0 could be decreased, as in case (1) above, and if $n > 1$, it is a consequence of the facts that $V_{n-1} \subset M_{n-1}$, $\pi_1(M_{n-1}) = 1$, and $\pi_1(V_{n-1}) \neq 1$.) Hence $H_1(V_{n-1})$ is infinite, by the result of Seifert mentioned above, and it follows that there is a covering transformation τ of M_n of infinite order such that $D_n \cap \tau(D_n) \neq \emptyset$. This in turn gives rise to a simple double curve in D_0, contrary to hypothesis.

So in the proofs of both the loop theorem and Dehn's lemma, covering spaces are used to select the switches that are to be performed, on C and D_0, respectively. As Papakyriakopoulos says of the proof of Dehn's lemma:

> Actually, looking closer at the proof of Dehn's lemma in [72], we observe that we actually construct the desired disc [72, ll. 34–38, p. 2], and that *the construction is carried out by means of successive cuts.*[9]

Arnold Shapiro suggested the use of 2-fold coverings instead of universal coverings in the tower construction ([74, p. 323]), and Shapiro and Whitehead gave a simplified proof of Dehn's lemma using such coverings [98]. (The advantage of using 2-fold coverings is that, under such a covering $\widetilde{M} \to M$, the image F in M of a nonsingular surface $\widetilde{F}$ in $\widetilde{M}$ will have only double points, and so switches may be performed on F without difficulty.) Finally, again using 2-fold coverings, Stallings gave a proof of the disk theorem, [103] which, as we have noted, combines the loop theorem and Dehn's lemma, and this is the statement that is normally used in practice.

In retrospect, with Stallings' proof of the disk theorem, we can see that the assumption in Dehn's lemma that the boundary of the disk is embedded is in a sense a red herring. On the other hand, it seems to have played an important metamathematical role in this whole development. For it led Papakyriakopoulos to consider the two statements, the loop theorem and Dehn's lemma, separately, and it was in trying to prove the former that the key idea of using covering spaces suggested itself to him, this in turn leading him to take a similar approach to the latter.

The version of the loop theorem for duality spaces proved by Casson and Gordon [17, Theorem 4.5], shows that there are also mathematical grounds for separating the loop theorem from Dehn's lemma. In that version, F is still a surface, but the 3-manifold M is replaced by any complex which satisfies 3-dimensional Poincaré–Lefschetz duality over some field of untwisted coefficients, emphasizing the essentially $2\frac{1}{2}$-dimensional character of the loop theorem.

Recently, an interesting and entirely new proof of the disk theorem has been given by Johannson (Klaus, not Ingebrigt), using hierarchies [54].

Let us now make some remarks on the triviality problem for knots: (how) can you decide whether or not a given knot is trivial? This question clearly lies behind Dehn's result that a knot K is trivial if and only if its group $\pi_1(S^3 - K)$ is isomorphic to $\mathbb{Z}$. Ignoring the fact that the proof uses Dehn's lemma, this statement "reduces" the triviality question for a knot to an algebraic question, namely, is its group Abelian? Although Dehn did, in fact, refer to this as a "solution" to the knot triviality problem, he was very much aware that it is not at all clear that the algebraic question is any easier, or if it can be solved at all. In fact, it was precisely this kind of question, involving fundamental groups of 2- and 3-dimensional manifolds, that led him to articulate and bring to the fore the word problem and isomorphism problem for finitely presented groups. (For a detailed account of the

[9] I.e. Dehn switches.

topological origins of combinatorial group theory, and in particular the influence of the work of Tietze and Dehn, see [18].)

With the proofs, in the 1950's, that there are finitely presented groups with unsolvable word problem, and that the isomorphism problem, or even the triviality problem, for finitely presented groups, is unsolvable, the equivalence of the knot triviality problem to a question about (apparently fairly complicated) finitely presented groups made it seem more likely to be unsolvable. The same held for other, more direct equivalences. For example, in his very nice popular article [112], Turing shows, by considering elementary moves on knots that lie on the unit lattice in $\mathbb{R}^3$, that the knot problem is equivalent to a problem about substitutions on strings of letters which does not seem to have much structure. Later, after listing some decision problems that have been shown to be unsolvable, he says:

> It has recently been announced from Russia that the 'word problem in groups' is not solvable. This is a decision problem not unlike the 'word problem in semi-groups', but very much more important, having applications in topology: attempts were being made to solve this decision problem before any such problems had been proved unsolvable. ... Another problem which mathematicians are very anxious to settle is known as 'the decision problem of the equivalence of manifolds'.... It is probably unsolvable, but has never been proved to be so.[10] *A similar decision problem which might well be unsolvable is the one concerning knots which has already been mentioned.*[11]

In this climate, it was therefore probably something of a shock when, at the International Congress of Mathematicians in Amsterdam in 1954, Haken gave a short address in which he announced that the triviality problem for knots was solvable, using his theory of normal surfaces [41]. The details of the proof appeared in 1961 [42]. Later, in the mid 1970's, the knot problem was also shown to be solvable; see [117].

We have seen that the Poincaré conjecture seems to have been the motivation for both Dehn's formulation of his lemma and Papakyriakopoulos' proof of it. It was also the problem that first led Whitehead into 3-dimensional topology, with his false proof of the conjecture in [119]. In fact this proof also implied that any contractible open 3-manifold is PL-homeomorphic to $\mathbb{R}^3$. But Whitehead soon realized his mistake, and came up with his famous counterexample [121] (an informal description is given in [120]). This *Whitehead manifold* W is defined to be $S^3 - \bigcap_{n=0}^{\infty} T_n$, where $T_0 \supset T_1 \supset \cdots$ is a certain nested sequence of solid tori, derived from the Whitehead link. Thus T_n is unknotted in S^3, and T_{n+1} is null-homotopic in T_n but does not lie in a 3-cell in T_n. It follows that $\pi_1(W) = 1$, $H_2(W) = 0$, and W is irreducible. However, Whitehead shows in [121] that W is not PL-homeomorphic to $\mathbb{R}^3$. In [68], the geometric arguments of [121] (recall that Dehn's lemma was not available) are replaced by algebraic arguments involving the fundamental group, and there it is proved that W is not even homeomorphic to $\mathbb{R}^3$.

But these influences of the Poincaré conjecture are somewhat indirect, and in many ways it has tended to become an isolated problem, with progress in 3-dimensional topology going on independently of it, although recently the work of Thurston [109], and in particular his geometrization conjecture, has put it in a broader context.

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[10] The unsolvability of the homeomorphism problem for manifolds of dimension $\geqslant 4$ was established by Markov in 1958 [61].

[11] My italics, C. McA. G.

12. π_2 and the sphere theorem

As homotopy theory arose and developed in the 1930's, with the work of Hopf and Hurewicz, investigations were begun on the homotopy properties of 3-manifolds. In 1936 Eilenberg [28] proved that if X is a nonseparating continuum in S^3, for example a knot, such that $\pi_1(S^3 - X) \cong \mathbb{Z}$, then $S^3 - X$ is aspherical (i.e. $\pi_i(S^3 - X) = 0, i \geqslant 2$).

(As an aside, it is interesting to see the terminology adjusting to the unavailability of Dehn's lemma. Eilenberg says:

> ... $\pi_1(S^3 - K) \cong \mathbb{Z}$, which means, in the sense of knot theory (based on the notion of the fundamental group), that K is unknotted.

Later, Whitehead [122] uses the term "ordinary circuit" to mean a knot that does not provide a counterexample to Dehn's lemma, i.e. one that is either unknotted or has the property that its latitude is not null-homotopic in its complement.)

Eilenberg also gave a necessary and sufficient condition for a 2-component link $K_1 \cup K_2$ to have the property that K_1 is a deformation retract of $S^3 - K_2$ (namely, $\pi_1(S^3 - K_2) \cong \mathbb{Z}$ and the linking number of K_1 and K_2 is 1). This led him to ask the following two questions, which turned out to be quite influential.

(1) For which knots K in S^3 is $S^3 - K$ aspherical?

(2) For which 2-component links L in S^3 is $S^3 - L$ aspherical?

Regarding his second question, Eilenberg notes that if L is a split link then $\pi_2(S^3 - L) \neq 0$. He also shows that if $\pi_1(S^3 - L) \cong \mathbb{Z} * \mathbb{Z}$ then $S^3 - L$ is not aspherical, for otherwise (by an earlier theorem of his) $S^3 - L$ would be deformable to a one-dimensional subcomplex, and hence would have 2nd Betti number equal to 0, contradicting Alexander duality. Finally, he shows (by considering the homology of the universal covering space) that a proper, connected, open subset U of S^3 is aspherical if and only if $\pi_2(U) = 0$. (This argument of course applies to any open 3-manifold.)

Inspired by Eilenberg's paper, Whitehead attacked the question of the asphericity of knot and link complements in his 1939 paper [123], (in which he thanks Eilenberg for many valuable suggestions). His approach is essentially algebraic. Starting with a knot K in S^3, he considers the cell decomposition of $S^3 - K$ corresponding to the Wirtinger presentation of $\pi_1(S^3 - K)$, obtaining a 2-complex X homotopy equivalent to $S^3 - K$. If $\widetilde{X}$ denotes the universal covering of X, then $\pi_2(X) \cong \pi_2(\widetilde{X}) \cong H_2(\widetilde{X})$, the last isomorphism being a consequence of the Hurewicz theorem. Moreover, $H_2(\widetilde{X}) \cong \ker \partial$, where $\partial : C_2(\widetilde{X}) \to C_1(\widetilde{X})$ is the boundary homomorphism.

Note also that $C_2(\widetilde{X})$ and $C_1(\widetilde{X})$ are the free $\mathbb{Z}\pi_1(X)$-modules on the 2-cells and 1-cells respectively of X. Thus $\pi_2(X) = 0$ if and only if ∂ is injective, and in this way the asphericity problem becomes equivalent to an assertion about a finite system of linear equations over $\mathbb{Z}\pi_1(X)$. Pointing out that this works for any graph in S^3, Whitehead, by explicit calculation, shows that the complements of the figure eight knot, the Whitehead link [122], and a certain knotted wedge of two circles, are all aspherical.

Next, Whitehead proves that if X_1, X_2 and $X_1 \cap X_2$ are aspherical polyhedra such that $\pi_1(X_1 \cap X_2) \to \pi_1(X_i)$ is injective, $i = 1, 2$, then $X = X_1 \cup X_2$ is aspherical. (He says the proof is mainly due to Eilenberg.) He uses this to show that the asphericity of the complement of a link L is preserved by doubling (in the sense of [122]) a component K of L, provided that $\pi_1(T) \to \pi_1(S^3 - L)$ is injective, where T is the boundary of a tubular neighborhood of K. Recalling that Eilenberg had remarked that the asphericity of

$S^3 - L$ reflects some sort of *linking* of the two components of L, Whitehead points out that his doubling construction shows that $S^3 - L$ may be aspherical even though the two components of L are only linked in a very weak sense, specifically, for any given n, L may be chosen so that the components are not *n-linked* in the sense of Eilenberg [29]. This observation leads him to ask:

> *If X is a closed subset of S^3, is $S^3 - X$ aspherical unless X is a disjoint union $X_1 \coprod X_2$, $X_1 \neq \emptyset \neq X_2$, where X_1 is contained in a 3-cell which does not meet X_2?*

He notes that this is equivalent to:

> *If U is an open subset of S^3, is $\pi_2(U) = 0$ provided every embedded S^2 in U bounds a 3-cell in U?*

Thus Whitehead has arrived at the right "conjecture" about the asphericity of submanifolds of S^3. As Papakyriakopoulos says in [74]:

> It was precisely this conjecture which stimulated the present author to prove during the summer of 1956 the following sphere theorem.

In 1947 Higman took up the asphericity question [46], and used Whitehead's algebraic formulation to show that if L is a link in S^3 such that $\pi_1(S^3 - L)$ is a nontrivial free product, then $\pi_2(S^3 - L) \neq 0$, generalizing Eilenberg's result mentioned above.

The only further progress on the question of the "asphericity of knots and links", until Papakyriakopoulos' complete solution in 1957, was Aumann's proof [10] that complements of alternating knots and links are aspherical. This goes as follows. Let D be a reduced, alternating, connected diagram of a knot or link K in S^3. Shading the complementary regions of the diagram alternately black and white, we see that it determines two spanning surfaces for K. Let F be one of these surfaces, and assume for convenience that F is non-orientable (in fact this can always be arranged if K is a knot and D has a nonzero number of crossings); the orientable case is similar. The surface F has a neighborhood X_1 (a twisted I-bundle over F), such that X_1 and $X_2 = \overline{S^3 - X_1}$ are handlebodies. Then X, the complement of an open neighborhood of K, can be expressed as $X_1 \cup X_2$, where $X_1 \cap X_2 = \widetilde{F}$ is the 2-fold orientable cover of F. Clearly, the map $\pi_1(\widetilde{F}) \to \pi_1(X_1)$ is injective, and Aumann shows, using the fact that D is reduced and alternating, that the map $\pi_1(\widetilde{F}) \to \pi_1(X_2)$ is also injective. The asphericity of X now follows from the result of Whitehead mentioned above.

Appearing as it did just before [72], Aumann's result was overshadowed by that of Papakyriakopoulos. However, it has a feature which was to emerge later as an important notion in knot theory. Namely, taking K to be a knot, one can show that F can be chosen so that its boundary is not a latitude of K, and so the incompressible surface $\widetilde{F}$ represents a nonzero *boundary slope* of K. The potential usefulness of incompressible surfaces with boundary in knot complements was emphasized by Neuwirth [69], and it was later proved by Culler and Shalen [20], using deep results on representations of knot groups in $PSL_2(\mathbb{C})$, that *every* (nontrivial) knot has a nonzero boundary slope. It turns out that the boundary slopes of a knot K play an important role in the study of the manifolds obtained by Dehn surgery on K.

Now we come to the sphere theorem, which asserts that if an orientable 3-manifold contains a singular homotopically essential 2-sphere then it contains a nonsingular one.

SPHERE THEOREM. *Let M be an orientable 3-manifold such that $\pi_2(M) \neq 0$. Then M contains an embedded 2-sphere which is not null-homotopic.*

Papakyriakopoulos proved a "conditional" version of the sphere theorem at the same time that he proved Dehn's lemma [72]. In fact, his proof of the former is modelled exactly on that of the latter, and it is because of this that he needs an extra hypothesis, to deal with the case where n, the height of the tower, is 1, and the sphere S_1 (which plays the role of the disk D_1 in the proof of Dehn's lemma) is nonsingular. To ensure the existence of a covering transformation τ of M_1 of infinite order such that $S_1 \cap \tau(S_1) \neq \emptyset$, he needs to assume that $H_1(V_0)$ is infinite. This follows as before if ∂V_0 does not consist of 2-spheres, but to take account of the possibility that it does, Papakyriakopoulos adds the hypothesis that M embeds in a 3-manifold N such that any nontrivial finitely generated subgroup of $\pi_1(N)$ has infinite commutator quotient group, (in other words, $\pi_1(N)$ is *locally indicable*). Since this condition is vacuously satisfied if $\pi_1(N) = 1$, Papakyriakopoulos' version is enough to prove the asphericity of knots; more generally, it proves Whitehead's conjecture characterizing the aspherical open subsets of S^3.

The additional hypothesis was soon shown to be unnecessary. Quoting Papakyriakopoulos [74, p. 319]:

> In October 1957 J.W. Milnor proved a more general sphere theorem. Finally in December 1957 J.H.C. Whitehead ... proved the sphere theorem in complete generality.

Whitehead [125] achieved this by making the following modifications to the definition of the tower. Firstly, the tower stops when $\pi_1(V_n)$ is finite, as opposed to trivial. Secondly, the coverings $M_i \to V_{i-1}$ are universal, as before, except for the first, $M_1 \to V_0$, which is defined to be that corresponding to the cyclic subgroup of $\pi_1(V_0)$ generated by a nontrivial covering transformation τ of the universal covering $\widetilde{V}_0 \to V_0$ such that $S_1 \cap \tau(S_1) \neq \emptyset$, where S_1 is a lift of the original 2-sphere $S_0 \subset V_0 \subset M$ to $\widetilde{V}_0$. Incidentally, the papers [98, 125] marked Whitehead's return to 3-dimensional topology after an absence of almost twenty years.

As we saw in Section 11, Shapiro and Whitehead, and Stallings, showed that 2-fold coverings could be used to considerably simplify the proofs of Dehn's lemma and the loop theorem, but, interestingly, this does not work for the sphere theorem. Perhaps it is for this reason that Stallings says in [104] that

> The proofs of Dehn's lemma and the Loop Theorem are an order of magnitude easier than is the proof of the Sphere Theorem.

The difference between the theorems is also reflected in Johannson's approach, using hierarchies. While this gives a proof of the disk theorem, in the case of the sphere theorem it merely reduces the problem to proving that if M is a closed, orientable, irreducible, non-Haken 3-manifold then $\pi_2(M) = 0$.

The sphere theorem is false for non-orientable 3-manifolds, as the example $\mathbb{RP}^2 \times S^1$ shows. However, Epstein, in his Cambridge Ph.D. dissertation (see [32]), showed that the following version, which he calls the projective plane theorem, holds without the assumption of orientability: if M is a 3-manifold such that $\pi_2(M) \neq 0$, then there is an essential map $S^2 \to M$ which either is an embedding or has image a 2-sided projective plane. In the case that M is non-orientable, this is proved by going to the 2-fold orientable cover $\widetilde{M}$ of M, taking an essential embedded 2-sphere S in $\widetilde{M}$ (whose existence is guaranteed by the sphere theorem), and doing a cut and paste argument on $S \cup \tau(S)$, where τ is the nontrivial covering transformation of $\widetilde{M}$. So here we see another application of 2-fold coverings in this context, similar to the ones we have already met.

Another line of development here is the relationship between $\pi_2(M)$ and the *ends* of $\pi_1(M)$. The theory of ends was initiated in Freudenthal's 1930 Berlin dissertation (see [37]), and the notion of the number of ends $e(G)$ of a group G was defined by Hopf [49], who proved that $e(G) = 0, 1, 2$, or ∞. In [101], Specker used this theory to show that for a closed 3-manifold M, $\pi_2(M)$ is determined by $\pi_1(M)$, a fact which had been announced without proof by Hopf [48]; more precisely, he showed that $\pi_2(M)$ is a free Abelian group of rank n, where $n = 0, 1, \infty$, according as $e(\pi_1(M))$ is less than 2, 2, or ∞. This is proved by applying Poincaré duality (using cohomology based on finite cochains) in the universal cover of M.

Applying similar considerations to 3-manifolds with boundary, Specker also showed that the asphericity of knots is equivalent to the assertion that the number of ends of a knot group is 1 or 2. In particular, since the group of a torus knot has an infinite cyclic center, it follows from a theorem of Hopf [49] that it has 1 or 2 ends; and, hence, that complements of torus knots are aspherical. But except in this special case, the equivalence established by Specker did not lead to progress on the asphericity question; rather, it was the other way round: after Papakyriakopoulos proved his sphere theorem he could deduce the fact about ends of knot groups. However, the direction of implication that no doubt Specker had in mind did eventually reappear, in the later work of Stallings [104].

Specker's paper also contains an application to the question of which Abelian groups can be fundamental groups of (compact) 3-manifolds. For closed, orientable 3-manifolds this was solved earlier by Reidemeister [87]. Reidemeister observes that for such a manifold M, $\pi_1(M)$ has a finite presentation with the same number of generators as relations, and shows that the only Abelian groups with this property are $\mathbb{Z}$, $\mathbb{Z}_n$ $(n \geqslant 1)$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}_n$ $(n \geqslant 2)$, and $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. The main part of the proof is now to rule out the possibilities $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}_n$, which is done by using duality in the cellular chains of the universal covering.

Specker considers manifolds with boundary, and shows that if M is a compact, orientable 3-manifold whose boundary is nonempty and does not consist entirely of 2-spheres, and whose fundamental group $\pi_1(M)$ is Abelian, then $\pi_1(M) \cong \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.

Finally, in [32] Epstein proves that the only finitely generated Abelian groups that can be subgroups of the fundamental group of any 3-manifold (not necessarily paracompact or orientable) are $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}_2$, and $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. (He also shows that if M is a compact non-orientable 3-manifold with $\pi_1(M)$ finite, then $\pi_1(M) \cong \mathbb{Z}_2$, and in fact M is homotopy equivalent to $\mathbb{RP}^2 \times I$ with a finite number of open 3-balls removed.)

The paper [126] of Whitehead is another which fits into this general end-theoretic context. Here he proves that if M is a 3-manifold, then $\pi_1(M)$ is isomorphic to $\mathbb{Z}$ or a nontrivial free product if and only if M contains an essential embedded 2-sphere. For the nontrivial implication, Whitehead notes that the hypothesis on $\pi_1(M)$ implies that the number of ends of $\pi_1(M)$ is 2 or ∞, and hence, by Specker [101], $\pi_2(M) \neq 0$. If M is orientable, the result is now a consequence of the sphere theorem. Whitehead shows that the statement also holds in the non-orientable case. However, this follows easily from Epstein's projective plane theorem.

We now go back to Kneser's 1929 paper [56]. In the last section of that paper he gives a proof of the following statement.

KNESER'S CONJECTURE. *If M is a 3-manifold such that $\pi(M) \cong G_1 * G_2$, then $M \cong M_1 \# M_2$, where $\pi_1(M_i) \cong G_i$, $i = 1, 2$.*

This is another manifestation of the principle that, in dimension 3, the fundamental group determines the topology.

Kneser's argument is hard to follow, but may be roughly summarized thus. First, a 2-dimensional spine of the 3-manifold M is modified to get two disjoint 2-complexes X_1, X_2 in M, with $\pi_1(X_i) \cong G_i$, $i = 1, 2$, whose inclusions into M induce the natural inclusions of the factors G_i into $G_1 * G_2 \cong \pi_1(M)$. Next one finds a closed surface F in M, separating X_1 and X_2, such that the map $\pi_1(F) \to \pi_1(M)$ is trivial. Using Kneser's Hilfsatz, F may be compressed to a disjoint union of 2-spheres. If two of the 2-spheres can be joined by a path that is null-homotopic in M, then $X = X_1 \coprod X_2$ can be changed so that it misses this path, and then the 2-spheres can be connected by a tube to form a single 2-sphere. Doing this as often as possible, Kneser argues that one must end up with a single 2-sphere, separating M into two components M_1', M_2' with $\pi_1(M_i') \cong G_i$, $i = 1, 2$.

It was soon after writing this paper that Kneser discovered the flaw in Dehn's proof of his lemma, and so he says, in a footnote added in proof, that because of this his proof should be considered incomplete. However, his argument is sufficiently unclear that even when the Hilfsatz was finally established, with the proof of the loop theorem and Dehn's lemma, his theorem was still not regarded as having been proved, and Papakyriakopoulos in [74] therefore termed it "Kneser's conjecture". Papakyriakopoulos envisages that one would approach this conjecture in two steps:

> This suggests that the gap between [the hypothesis] and the conclusion of Kneser's conjecture is so great that it has to be factored, and we first have to prove that [the hypothesis] implies $\pi_2 \neq 0$, and then that $\pi_2 \neq 0$ implies the desired conclusion. It seems that the first step has to be proved by *algebraic topological* techniques, and the second one by using the sphere theorem and *something more*, because the sphere theorem is not enough to provide us with the conclusions of Kneser's conjecture. Thinking now that the algebraic topological techniques were rather undeveloped in 1928, we easily conclude that it was rather hopeless, to expect to have a satisfying proof of this strong statement at that time.

Nevertheless, Stallings, to whom Papakyriakopoulos had suggested the problem, gave a proof of Kneser's conjecture, in his 1959 Princeton Ph.D. thesis [102], which did not follow this scheme, but which was, in outline, very much along the lines indicated by Kneser.

In particular, Stallings' proof did not use the sphere theorem. Note, however, that the conclusion clearly implies (if the groups G_1 and G_2 are nontrivial) that M contains an embedded essential 2-sphere. This leads us to Stallings' work of about ten years later, which brings these ideas about π_2 and ends of π_1 to a full circle. The key is Stallings' result [104] that a finitely generated group G has $e(G) \geqslant 2$ if and only if G is a nontrivial free product with amalgamation $A *_F B$, or an HNN-extension $A*_F$, where the amalgamating subgroup F is finite. Now let M be a (say, closed, orientable) 3-manifold with $\pi_2(M) \neq 0$. Applying the Hurewicz theorem and Poincaré duality to the universal covering of M, it follows easily that $e(\pi_1(M)) \geqslant 2$. Hence, $G = \pi_1(M)$ splits as described above over a finite group F. Constructing a $K(G, 1)$ space K_G containing a bicollared copy of a $K(F, 1)$ space K_F, we have a map $f : M \to K_G$ inducing an isomorphism on fundamental groups, and we may assume by transversality that $f^{-1}(K_F)$ is a 2-sided surface S in M. Using the disk theorem, the map f may be homotoped so that for each component S_0 of

S, $\pi_1(S_0) \to \pi_1(M)$ is injective. Since $\pi_1(K_F) \cong F$ is finite, this implies that S_0 is a 2-sphere. Since null-homotopic components S_0 may be eliminated by a further homotopy of f, Stallings concludes that M contains an essential embedded 2-sphere.

This work demonstrates a deep relationship between the fundamental group of a 3-manifold and its topology, and indeed Stallings sees the connection between group theory and 3-dimensional topology in even broader terms. In [104, pp. 1, 2] he makes the following remarks, which have been vindicated by recent work in geometric group theory:

> Philosophically speaking, the depth and beauty of 3-manifold theory is, it seems to me, mainly due to the fact that its theorems have off-shoots that eventually blossom in a different subject, namely group theory. Thus I tend to believe that new results in the theory, such as Waldhausen's [115], may eventually have relatives in group theory; the solution of the Poincaré Conjecture [81], if it ever occurs, will have group-theoretic consequences of a wider nature.

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