Please let me know if any of the problems are unclear or have typos. Please let me know if you have suggestions for exercises. For some of the problems I have given a (very vague) level of difficulty.

Exercise 2.1. Suppose that M, N, and P are compact, connected surfaces. Verify the following properties of the connect sum.

a) $M \# S^2 \cong M$

b) $M \# N \cong N \# M$

c) $(M \# N) \# P \cong M \# (N \# P)$

[You may simplify the exercise by assuming that the surfaces are oriented.]

Exercise 2.2. Suppose that M and N are compact, connected surfaces. We denote the connect sum of n copies of M by nM. Verify the following properties of the connect sum.

a) $K^2 \cong 2P^2$

b) $T^2 \# P^2 \cong 3P^2$

c) $\chi(M \# N) = \chi(M) + \chi(N) - 2$

Exercise 2.3. Equip S^3 with the round metric. Prove that the induced homomorphism from $O(4)$ to $\mathrm{Isom}(S^3)$ is an isomorphism.

Exercise 2.4. [Medium.] Suppose that $M = M^3$ is a connected elliptic manifold. (That is, M admits a complete (G, S^3) structure where G is the pseudo-group obtained by restricting elements of $\mathrm{Isom}(S^3)$.) Give a direct proof that M is a manifold quotient of S^3 by some finite subgroup $\Gamma < \mathrm{Isom}^+(S^3)$. Deduce that M is orientable.

Exercise 2.5. Let $L(p, q)$ denote the (p, q)–lens space. Let S^k and P^k denote the sphere and the projective space, respectively. Let $\mathrm{SU}(n)$ denote the special unitary group. Let $\mathrm{UT}(M)$ denote the unit tangent bundle to the manifold M. Verify the following homeomorphisms.

a) $L(1, 1) \cong S^3 \cong \mathrm{SU}(2)$

b) $L(2, 1) \cong P^3 \cong \mathrm{SO}(3) \cong \mathrm{UT}(S^2) \cong \mathrm{Isom}^+(S^2)$

c) $L(4, 1) \cong \mathrm{UT}(P^2)$

Exercise 2.6. Suppose that $L = L(p, q)$ and $L' = L(p', q')$ are lens spaces.

a) Suppose that $L \cong L'$. Prove that $p = p'$.

b) Give necessary and sufficient conditions on q and q' to ensure that $L \cong L'$.

c) [Hard.] Give necessary and sufficient conditions on q and q' to ensure that L is homotopy equivalent to L'.

Exercise 2.7. Suppose that $L = L(p, q)$ is a lens space equipped with its usual round metric. Fix $x \in L$. We define $\mathcal{D}_x \subset L$, the *Dirichlet domain* about x, as follows.

$$\mathcal{D}_x = \{y \in L \mid \text{there is a unique shortest path from } y \text{ to } x\}$$

The complement $\mathcal{C}_x = L - \mathcal{D}_x$ is called the *cut locus*.

a) Find a point $x \in L$ so that $\mathcal{D}_x$ is a *lens*: a three-ball bounded by two spherical caps.

b) [Hard.] Describe, for generic $x \in L$, the combinatorics of $\mathcal{D}_x$.

c) [Open.] Suppose that $x \in L$ is generic. The dual of the Dirichlet domain $\mathcal{D}_x$, denoted $\mathcal{T}_x$, is the *Delaunay triangulation* at x. Show that $\mathcal{T}_x$ is *minimal* in the following sense: it has the smallest number of tetrahedra amongst all combinatorial triangulations of L.

Exercise 2.8. [Medium.] Any elliptic manifold is covered (at most 60–fold) by a lens space.

Exercise 2.9. Let D be a dodecahedron. For each $k = 1, 3, 5$, we can form a topological space M_k by gluing opposite faces of D with a $2\pi k/10$ twist. For each k,

a) verify that M_k is a three-manifold,

b) give a presentation of $\pi_1(M_k)$, and

c) realise M_k as a geometric manifold.

Exercise 2.10. Give an explicit example of a (pseudo-)Anosov map $f : F \to F$. Sketch its singular foliations; compute its stretch factor $\lambda = \lambda_f$.