Please let me know if any of the problems are unclear or have typos. Please let me know if you have suggestions for exercises. For some of the problems I have given a (very vague) level of difficulty. Finally, if you want to do just part of a problem, let me know.

Exercise 3.1. Find all covering maps amongst the seven manifolds with $S^2 \times \mathbb{R}$ geometry.

Exercise 3.2. Fix a commutative ring R, with identity. The *Heisenberg group* over R, denoted $H(R)$, is the group of three-by-three upper triangular matrices with ones on the diagonal and elements of R above the diagonal. Show that the torus bundle with monodromy

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is homeomorphic to the quotient $H(\mathbb{R})/H(\mathbb{Z})$.

Exercise 3.3. Prove that $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{R})$.

Exercise 3.4. [Hard.] Suppose that M is a closed, connected three-manifold with $\mathbb{H}^2 \times \mathbb{R}$ geometry. Prove that there is

- a closed, connected, oriented surface F,
- a periodic homeomorphism $f\colon F \to F$, and
- a finite cover M' of M (of degree at most four)

so that M' is homeomorphic to the surface bundle M_f.

Exercise 3.5. [Medium.] Suppose that M is a closed, connected, oriented three-manifold. Suppose that $\mathcal{F}$ is a one-dimensional foilation of M where all leaves are circles. Prove that for every leaf $\ell \in \mathcal{F}$ there is

- a pair of integers p, q and
- a neighbourhood $V = V(\ell)$

so that $(V, \mathcal{F}|V)$ is homeomorphic to the foliated solid torus $V_{p,q}$

We call ℓ a *critical* leaf if $p > 1$. Prove that $\mathcal{F}$ has only finitely many critical leaves.

Exercise 3.6. Prove that $\mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{UT}(\mathbb{H}^2) \cong \mathrm{interior}(D^2) \times S^1$.

Exercise 3.7. [Hard.] Prove that the following manifolds are homeomorphic.

1. The trefoil knot exterior $X_T = S^3 - T$.
2. The surface bundle with fiber a once-punctured torus $S_{1,1}$ and with monodromy

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

3. $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SL}(2,\mathbb{Z})$.
4. The unit tangent bundle to the hyperbolic orbifold $S^2(2,3,\infty)$.

Finally, show that X_T is a deformation retract of $\mathbb{C}^2 - \{z^2 = w^3\}$.

Exercise 3.8. [Hard.] Prove that the following manifolds are homeomorphic.

1. The figure-eight knot exterior $X_K = S^3 - K$.
2. The surface bundle with fiber a once-punctured hexagonal torus $S_{1,1}$ and with monodromy
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$
3. The ideally triangulated manifold shown in Figure 3.9.

Finally, show that X_K is a twelve-fold cover of $\mathbb{H}^3/\,\mathrm{PSL}(2,\mathbb{Z}[\omega])$. Here ω is a primitve sixth root of unity.

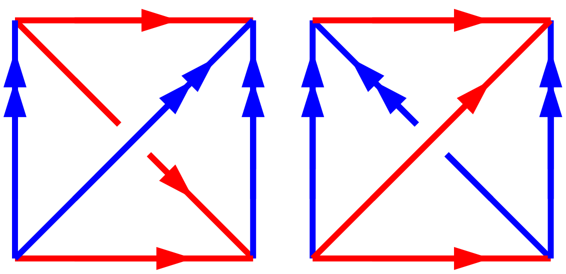

Figure 3.9: Two ideal tetrahedra, with face pairings as indicated by the arrowed edges.

Exercise 3.10. Suppose that M is a three-manifold with boundary. Let $B = \mathbb{B}^3$ be a copy of the three-ball. Fix closed disks $D \subset \partial M$ and $E \subset \partial B$ as well as a homeomorphism $\phi\colon D \to E$. Prove that $M \cup_\phi B$, the *boundary connect sum*, is homeomorphic to M. [You will need the fact that ∂M has a *collar neighbourhood* $\partial M \times I \subset M$.]