Please let me know if any of the problems are unclear or have typos. Also, do let me know if you have suggestions for exercises. For some of the problems I have given a (very vague) level of difficulty. Finally, if you want to do just part of a problem, let me know.

Exercise 7.1. Suppose that M is a compact connected oriented three-manifold, not homeomorphic to the three-sphere. Let M_n be the manifold obtained by taking the connect sum of of n copies of M. Prove that $c(M_n)$ (the Matveev complexity) is bounded above and below by linear functions of n.

Exercise 7.2. Suppose that S_g is a closed connected oriented surface of genus g. Set $M_g = S_g \times S^1$.

- Prove that M_g is irreducible.
- Prove that $c(M_g)$ is bounded above and below by linear functions of g.

Exercise 7.3. [Hard.] We call a three-manifold M an *integral homology three-sphere* if the homology groups of M, over $\mathbb{Z}$, are isomorphic to those of S^3. We call a three-manifold M *atoroidal* if its fundamental group has no subgroups ismorphic to $\mathbb{Z}^2$.

Give an example of a sequence of closed, connected, oriented manifolds M_n so that

- each M_n is irreducible and atoroidal,
- each M_n is an integral homology three-sphere, yet
- the Matveev complexity $c(M_n)$ grow linearly with n.

Exercise 7.4. Suppose that M is a closed, connected, oriented three-manifold. Suppose that S is a closed, connected, transversely oriented, embedded surface in M. Suppose that γ is a closed, connected, oriented, embedded loop in M. We define the *algebraic intersection* number $\langle S, \gamma \rangle$ as follows: isotope S to be transverse to γ and count the points of $S \cap \gamma$ with sign.

- Show that $\langle S, \gamma \rangle$ depends only on the homology classes of γ and S.

Fix M and S as above. Define the function $\sigma \colon H_1(M; \mathbb{Z}) \to \mathbb{Z}$ by $\sigma([\gamma]) = \langle S, \gamma \rangle$.

- Show that σ is well-defined.
- Show that if σ is non-zero then it is surjective. [Hint: we have assumed that S is connected.]
- Suppose that σ is non-zero. Show that, for any triangulation K of M, the homology class $[S]$ contains a normal surface.

Exercise 7.5. Define the map $f_n \colon \mathbb{C} \to \mathbb{C} \times \mathbb{R}$ by $f_n(z) = (z^n, \mathrm{Imag}(z))$.

- [Easy.] Sketch the image of f_2. This is the local model for a *simple branch point*.
- Sketch the image of f_n. This is a local model for an *n–fold branch point*.
- Suppose instead that the first coordinate is $z^n - n\epsilon^{n-1}z$ for ϵ real and very small. Sketch the image; count the number and kind of its branch points.