Please let me know if any of the problems are unclear or have typos. Also, please let me know if you have suggestions for exercises.

Exercise 9.1. Suppose that (M, K) is a closed, connected, triangulated three-manifold. Suppose that $f\colon S \to (M, K)$ is a PL minimal surface. Recall that $\Gamma(f) \subset S$ is the preimage of $K^{(2)}$. Define $\Sigma(f) \subset S$ be the "locus of non-injectivity": that is, the set of points x in S so that there is some y in S with $y \neq x$ yet $f(y) = f(x)$. Prove the following.

- If $\Sigma(f) = S$ then f is a covering map of its image.
- Suppose that $\Sigma(f)$ is non-empty, but is not all of S. Then there is a vertex a, with adjacent edge e, of $\Gamma(f)$ so that a lies in $\Sigma(f)$ but e does not.

Exercise 9.2. [Half lives, half dies.] Suppose that M is a compact, connected, oriented three-manifold. Let $\iota\colon \partial M \to M$ be the inclusion map. Prove that the kernel of $\iota_*\colon H_1(\partial M) \to H_1(M)$ has rank one-half that of $H_1(\partial M)$.

Exercise 9.3. Suppose that M is a compact, connected, simply-connected three-manifold. Prove that all components of ∂M are two-spheres.

Exercise 9.4. Suppose that M is a compact, connected, simply-connected three-manifold. We orient M and give the components of ∂M their induced orientations.

- Prove that the (homotopy classes of the) components of ∂M generate $\pi_2(M)$.
- Prove that the sum of the (homotopy classes of the) components of ∂M are zero in $\pi_2(M)$.

Exercise 9.5. Supose that $(f_n\colon S^2 \to N_k \subset M_k)$ is a tower as in the proof of the sphere theorem. Let $\Sigma_k = \Sigma(f_k)$. Verify the following steps of the proof.

- For all k we have $\Sigma_{k+1} \subset \Sigma_k$.
- If $\Sigma_{k+1} = \Sigma_k$ then $N_{k+1} \to N_k$ is a homotopy equivalence.
- The tower is finite.