THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: MAY 2020

COHOMOLOGY AND POINCARÉ DUALITY

Time Allowed: **3 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2020' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER COMPULSORY QUESTION 1 AND TWO FURTHER QUESTIONS out of the four optional questions 2, 3, 4 and 5.

On completion of the assessment, you must upload your answer to Moodle as a single PDF document if possible, although multiple files (2 or 3) are permitted. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2020' webpage.

You must not upload answers to more than 3 questions, including Question 1. If you do, you will only be given credit for your Question 1 and the first two other answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question. The compulsory question is worth twice the number of marks of each optional question. Note that the marks do not sum to 100.

COMPULSORY QUESTION

1. Suppose that R is a commutative ring with unit 1_R.

a) Give a careful definition of a *category* $\mathcal{C}$. **[3]**

b) Define the category Mod_R of R–modules and check the axioms. **[3]**

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are categories.

c) Give a careful definition of a (covariant) *functor* $F\colon \mathcal{C} \to \mathcal{D}$. **[3]**

d) Define $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ by $F(A) = \mathrm{Hom}_R(R, A)$. Extend F to morphisms and prove that F is a functor. **[7]**

e) Suppose that $F, G\colon \mathcal{C} \to \mathcal{D}$ are functors. Give a careful definition of a *natural transformation* $\mu\colon F \to G$. **[3]**

f) Define Id to be the identity functor on the category Mod_R. Define $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ as in part (d). Show that there is a natural transformation $\mu\colon \mathrm{Id} \to F$. **[7]**

g) Let $\mathrm{Pairs}_{\mathrm{top}}$ be the category of pairs of topological spaces. Let $\mathrm{Kom}_{\mathrm{ab}}$ be the category of chain complexes of abelian groups. Briefly describe

$$\mathcal{L}\colon \mathrm{Pairs}_{\mathrm{top}} \to \mathrm{Kom}_{\mathrm{ab}}$$

the *long exact sequence* for the relative homology of a pair. **[6]**

h) With notation as in part (g): prove that the connecting homomorphism appearing in $\mathcal{L}$ is a natural transformation. **[6]**

i) With notation as in parts (g) and (h): suppose $f\colon (X, A) \to (Y, B)$ is a map of pairs. Prove that $\mathcal{L}(f)$ is a chain map. **[2]**

OPTIONAL QUESTIONS

2. Set $X = S^2 \times S^4$ and $Y = \mathbb{CP}^3$.

a) Give CW–structures on both X and Y; explain. **[8]**

b) Show that X and Y are compact and connected. **[2]**

c) Show that X and Y are simply connected. **[2]**

d) Compute the modules $H^k(X;\mathbb{Z})$ and $H^k(Y;\mathbb{Z})$ for all k. **[4]**

e) Describe the cup product structure on $H^*(X;\mathbb{Z})$ and $H^*(Y;\mathbb{Z})$. Prove that they are not isomorphic as rings (and thus X and Y are not homeomorphic as spaces). **[4]**

 Question 3 continued overleaf

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3. Let $\Pi\colon \mathbb{R} \to S^1 \subset \mathbb{C}$ be the universal covering defined by $\Pi(t) = \exp(2\pi i t)$. As a bit of notation we write $\Delta^1 = [v_0, v_1]$ for the standard one-simplex.

a) Suppose that $\sigma\colon \Delta^1 \to S^1$ is a singular one-simplex. Define what it means for a singular simplex $\widetilde{\sigma}\colon \Delta^1 \to \mathbb{R}$ to be a *lift* of σ with respect to Π. **[2]**

b) For any singular one-simplex σ, choose a lift $\widetilde{\sigma}$ and define the *winding number* of σ to be

$$\omega(\sigma) = \widetilde{\sigma}(v_1) - \widetilde{\sigma}(v_0)$$

Prove that the winding number is well-defined. **[4]**

c) Extend ω linearly to all one-chains; the result is the *winding cochain* $\omega \in C^1(S^1; \mathbb{R})$. Prove that ω is a cocycle. **[4]**

d) Prove that the winding cocycle ω is not a coboundary. **[5]**

e) Suppose that $z \in Z_1(S^1)$ is a singular one-cycle. Prove that $\omega(z)$ is an integer. **[5]**

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4. Suppose that U and V are copies of the two-torus $T^2 = S^1 \times S^1$. Fix homeomorphisms $u\colon T^2 \to U$ and $v\colon T^2 \to V$. Let $C \subset T^2$ be the curve $S^1 \times \{1\}$. Build a quotient space X from the disjoint union $U \sqcup V$ by gluing the curve $u(C)$ to the curve $v(C)$. To be precise, define

$$X = U \sqcup V \Big/ u(z) \sim v(z) \text{ for all } z \in C$$

a) Give a Δ–complex structure on X. **[4]**

b) Compute the Euler characteristic of X. **[2]**

c) Compute the modules $H^k(X; \mathbb{Z})$ for all k. **[6]**

d) Describe the cup product structure on $H^*(X; \mathbb{Z})$. **[8]**

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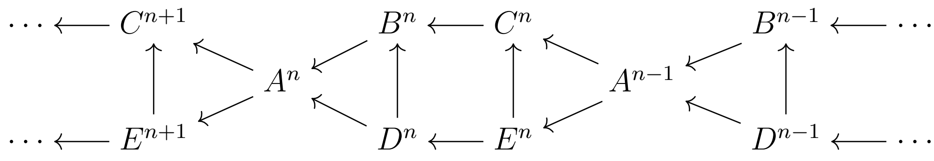
5. a) Consider the commutative diagram



where the two sequences across the top and bottom are exact. This diagram induces an exact sequence

$$\cdots \longleftarrow E^{n+1} \longleftarrow B^n \xleftarrow{\psi} C^n \oplus D^n \xleftarrow{\phi} E^n \longleftarrow B^{n-1} \longleftarrow \cdots$$

with maps obtained from the corresponding maps in the previous diagram, except that ϕ has a minus sign in one coordinate. Prove that $\mathrm{Ker}(\psi) = \mathrm{Im}(\phi)$. **[8]**

b) Suppose that R is a commutative ring with unit 1_R. Give the definition of a cohomology theory for CW–pairs. Carefully state the four axioms for homotopy invariance, exactness, excision, and disjoint union. **[8]**

c) Suppose that h^* is a cohomology theory. As a bit of notation, set $h^n(Y) = h^n(Y, \emptyset)$. The *Mayer-Vietoris axiom* for h is the following.

Suppose that (X, U) and (X, V) are CW-pairs, with $X = U \cup V$. Then there is a long exact sequence

$$\cdots \leftarrow h^{n+1}(X) \leftarrow h^n(U \cap V) \leftarrow h^n(U) \oplus h^n(V) \leftarrow h^n(X) \leftarrow h^{n-1}(U \cap V) \leftarrow \cdots$$

with all maps induced by inclusions or provided by the axioms for h^*.

Prove that the Mayer-Vietoris axiom follows from the axioms given in part (b). **[4]**
