THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: SUMMER 2022

COHOMOLOGY AND POINCARÉ DUALITY

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER ALL 4 QUESTIONS.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. Suppose that R is a commutative ring with identity 1_R which is, additionally, a principal ideal domain. Suppose that M, N, and Q are R–modules.

a) Define what it means for a chain complex $F_* = (F_k, \partial_k)$ to be a *free resolution* of M. **[2]**

b) Taking $R = \mathbb{Z}$, find free resolutions for the following $\mathbb{Z}$–modules.

(i) $M = \mathbb{Z}$. **[2]**

(ii) $M = \mathbb{Z}/n\mathbb{Z}$, for $n > 0$. **[2]**

(iii) $M = \mathbb{Q}$. **[2]**

c) Again, take R to be a commutative ring with identity 1_R which is, additionally, a principal ideal domain. Define the R–modules $\mathrm{Ext}^k_R(M, Q)$. **[2]**

d) Prove that $\mathrm{Ext}^{-1}_R(M, Q)$ is trivial. **[3]**

e) Compute $\mathrm{Ext}^0_R(M, Q)$. **[3]**

f) Prove the following isomorphisms.

(i) $\mathrm{Ext}^1_R(M \oplus N, Q) \cong \mathrm{Ext}^1_R(M, Q) \oplus \mathrm{Ext}^1_R(N, Q)$. **[3]**

(ii) $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, Q) \cong 0$. **[3]**

(iii) $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, Q) \cong Q/nQ$, for $n > 0$. **[3]**

2. Suppose that R is a commutative ring with identity 1_R. Suppose that X and Y are topological spaces. Throughout this problem all dual and cohomology modules have R as their coefficient module.

a) Define the cup product $\smile\colon C^k(X) \times C^\ell(X) \to C^{k+\ell}(X)$ directly in terms of singular chains and cochains. **[5]**

b) Briefly define the cohomology ring $H^*(X)$. Give (without proof) the cohomology ring in the special case where $X = \mathbb{CP}^3$ and $R = \mathbb{Z}$. **[5]**

c) Define the cross product $\times\colon H^*(X) \times H^*(Y) \to H^*(X \times Y)$. **[5]**

d) State the Künneth formula for cohomology. **[5]**

e) Taking $X = Y$ we have a cross product $\times\colon H^*(X) \times H^*(X) \to H^*(X \times X)$. Suppose that $\Delta\colon X \to X \times X$ is given by $\Delta(x) = (x, x)$. Suppose that α and β are cohomology classes in $H^*(X)$. Prove that $\Delta^*(\alpha \times \beta) = \alpha \smile \beta$. **[5]**

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3. In this problem we take $R = \mathbb{Z}$ to be our coefficient ring. Throughout we will refer to the Δ–complex structure, with the given labelling, on the two-torus T shown in Figure 1.

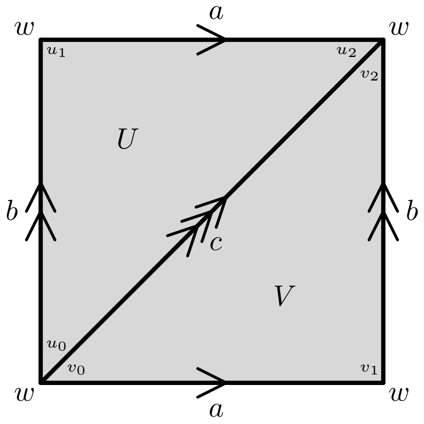

Figure 1: The arrows on the one-simplices a, b, and c both indicate their identifications and their orientations. These also induce the desired orientations on the two-simplices U and V. Note also that there is exactly one zero-simplex w.

a) Give the associated chain complex $C_*(T;\mathbb{Z})$. Include the matrices for the boundary homomorphisms ∂_k, as induced by the Δ–complex structure show in Figure 1. Use these to find the homology groups $H_k(T;\mathbb{Z})$ as well as explicit generators for each; you do not need to justify your work. **[5]**

For each k–simplex σ in Figure 1, we use $\sigma^* \in C^k(T;\mathbb{Z})$ to denote the cochain having $\sigma^*(\sigma) = 1$ and vanishing on all other k–simplices.

b) Give the associated cochain complex $C^*(T;\mathbb{Z})$. Include matrices for the coboundary homomorphisms δ_k. Use these to find the cohomology groups $H^k(T;\mathbb{Z})$ as well as explicit generators for each; justify your work. **[10]**

c) For every (ordered) pair of generators given part (b) find their cup product directly from the definitions. Organise your solutions in a (clearly labelled) "multiplication table"; justify your work. **[10]**

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4. In this problem we take $R = \mathbb{Z}$ to be our coefficient ring, equipped with the discrete topology. Suppose that M is an n–manifold.

a) Give the definition of the *homology bundle* $M_{\mathbb{Z}} \to M$. **[4]**

b) Give the definition of a $\mathbb{Z}$–*orientation* of M. **[2]**

c) Suppose that $M = M^2$ is the *Möbius band*: the quotient of the square $[0,1] \times (0,1)$ by the gluing $(1, y) \sim (0, 1 - y)$. Prove that the Möbius band does not admit a $\mathbb{Z}$–orientation. **[6]**

Suppose that $M = M^3$ is a compact, connected three-manifold without boundary. Suppose that the fundamental group $\pi_1(M)$ has the presentation

$$\pi_1(M) \cong \langle a, b \mid a^4 b a^{-1} b, a b^{-1} a b^2 \rangle$$

d) Show that $\pi_1^{\mathrm{ab}}(M)$, the abelianisation of $\pi_1(M)$, is the trivial group. **[3]**

e) Prove that $\pi_1(M)$ has no subgroups of index two. **[3]**

f) Prove that M is $\mathbb{Z}$–orientable. **[3]**

g) Find $H_k(M)$ for all k. Briefly justify your answers. [You may use, without proof, the fact that $\pi_1^{\mathrm{ab}} \cong H_1$.] **[4]**
