Cohomology + PD

One person designated to take notes each lecture. Lectures will be available on lecture capture where possible.

Q: Will we have exercise sheets?
A: Yes, first one available now, due Fri week 2. Exercises will be based on lecture content. 15% of module from 5 assignments.

Q: Support class?
A: Yes, probably starting next week.

Module is based around Hatcher's book.

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① Overview
Cohomology is a 'deep' module - it has many prerequisites.

② An example
Take $\mathbb{R}^n$ - an n-space and $\mathbb{C}^n$ - a complex n-space.
Also $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$

$$B^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\} \text{ (closed unit ball)}$$

$$\mathbb{RP}^n = \frac{\mathbb{R}^{n+1} - \{0\}}{\mathbb{R} - \{0\}} = \frac{\mathbb{R}^{n+1} - \{0\}}{x \sim \lambda x \text{ for } \lambda \in \mathbb{R} - \{0\}}$$

- the real projective space.

$\mathbb{C}P^n = \dfrac{\mathbb{C}^{n+1} - \{0\}}{\mathbb{C} - \{0\}}$ – the complex proj space

Exercise: Set $X = S^2 \times S^4$.
Set $Y = \mathbb{C}P^3$
Show that $X \not\cong Y$ (not homeomorphic)

This is a carefully chosen example to get rid of incorrect assumptions.

Important: These have the same homology groups in all dimensions.

Exercise: Give CW-structures for X and Y, as above.

Example: Set $Z = S^2 \vee S^4 \vee S^6$
(the one-point union)

A cartoon of Z:

Note this is not a manifold.

Remark: Z is not homeomorphic to X or Y
Is it homotopy equivalent?
Answer: <u>No</u>

③ (condensed overview)
Cohomology, The Universal Coefficient Thm,
+
④ Cup product, Poincaré duality.

⑤ Category Theory (a pleasant review)

Def: A category $\mathcal{C}$ is a collection of objects $Ob(\mathcal{C})$ and for every $X, Y \in Ob(\mathcal{C})$ a collection of morphisms $Mor(X, Y)$ is as follows:

For all $X, Y, Z \in Ob(\mathcal{C})$ there is

$$\circ_{X,Y,Z} : Mor(Y,Z) \times Mor(X,Y) \to Mor(X,Z)$$
$$(f,g) \longmapsto f \circ g$$

s.t. $(f \circ g) \circ h = f \circ (g \circ h)$ and for all X, there is $Id_X \in Mor(X, X)$ so that $f \circ Id_X = f$, $Id_X \circ g = g$

Q: Is this identity function unique?
A: Yes, If instead there were two identities, we can present a contradiction by composing them.

Exercise: If G is a group, then $\mathcal{C}_G = (pt, G)$ is a category.

Example: The path groupoid $\mathcal{G}$ of X (a top space) has $Ob(\mathcal{G}) = X$ and
$Mor(X, Y) = \{$homotopy classes of paths $X \to Y\}$

Here, $[\alpha]\circ[\beta] = [\alpha * \beta]$ where α, β are paths, and $\alpha * \beta$ is concatenation.

Notation:
We use letters like $\mathcal{C}, \mathcal{D}$ for general categories

eg. $\underline{Set}$, $\underline{Vec_{\mathbb{R}}}$, $(\underline{Mod_{\mathbb{Z}}})$, $\underline{Top}$

$\underline{Set}$ = (sets, functions)
$\underline{Vec_{\mathbb{R}}}$ = ($\mathbb{R}$-vector spaces, linear transformations)
$\underline{Mod_{\mathbb{Z}}}$ to be defined later
$\underline{Top}$ = (topological spaces, cts maps)

$\underline{Pairs}$ = ((X, A) s.t. $A \subset X$ are top. spaces, cts maps of pairs)

Here a 'map of pairs' is
$$f : (X, A) \longrightarrow (Y, B)$$
with $f : X \to Y$ cts and $f(A) \subset B$

Example: $\underline{Top_0}$ = (Pointed spaces (X, x_0) and pointed maps)

⑥ Functors:
Definition: A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ (categories)

is a map $F_{ob} : Ob(\mathcal{C}) \to Ob(\mathcal{D})$
and $\forall X, Y \in Ob(\mathcal{C})$ a map
$$F_{X,Y} : Mor(X, Y) \longrightarrow Mor(F(X), F(Y))$$
so that $F(f \circ g) = F(f) \circ F(g)$

(definition continued)

and $F(Id_X) = Id_{F(X)}$

Why is this necessary?

Example with ~~for~~ a group:

$$\mathbb{R}^* \longrightarrow Mat_2(\mathbb{R})$$
$$t \longmapsto \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$$

Here the identity does not map to itself.

Example: $\pi_1 : \underline{Top_\bullet} \longrightarrow \underline{Groups}$

$H_1 : \underline{Top} \longrightarrow \underline{Mod_{\mathbb{Z}}}$

where $\underline{Mod_{\mathbb{Z}}} = \underline{Abelian}$

Non-example:

Suppose G is a group.

Define $Z(G) = \underline{\text{centre}}$ of G

$= \{ x \in G \mid \forall\, y \in G, \; yx = xy \}$

Claim: The map $Z : \underline{Group} \longrightarrow \underline{Abelian}$ is not a functor.