Lecture 2

Tuesday, 11 January 2022 10:03 AM

Functors

Axioms for functors

1. $F(f \circ g) = F(f) \circ F(g)$
2. $F(Id_X) = Id_{F(X)}$.

Exercise: Show that neither axiom follows from the other.

Def: Let $X, Y \in Ob(\mathcal{C})$, a morphism $f : X \to Y$ is $\begin{cases} \text{monomorphism} & (\text{monic, mono}) \\ \text{epimorphism} & (\text{epic, epi}) \\ \text{isomorphism} & (\text{isomic, iso}) \end{cases}$

iff f has a $\begin{cases} \text{left} \\ \text{right} \\ \text{both} \end{cases}$ inverse.

Exercise: Functors preserve monic, epic, isomic morphisms.

Eg: Fix $k \in \mathbb{N}$, define

$$F : \underline{Pairs} \longrightarrow \underline{Ab} = \underline{Mod_{\mathbb{Z}}}$$
$$(X, A) \longmapsto H_k(X, A).$$

$$G : \underline{Pairs} \longrightarrow \underline{Ab}$$
$$(X, A) \longmapsto H_{k-1}(A)$$

Note both $F(f)$, $G(f)$ are the induced maps in both cases.

i.e. $F(f)[z] = [f(z)]$ "class of $f(z)$"

Not simply using f_* as ambiguous which induced map that would refer to.

Exercise: F and G are functors.

More examples:

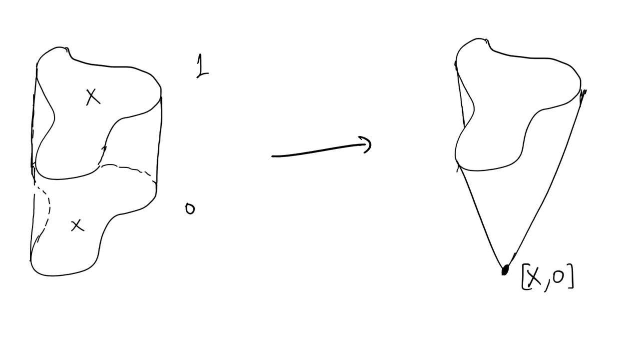
Eg: Forgetful functors

$\underline{Vec_k} \longrightarrow \underline{Set}$

k is a field henceforth

Eg: Topological functors

$$C : \underline{Top} \longrightarrow \underline{Top_*}$$
$$X \longmapsto (CX, (x,0)) \quad \text{where } CX = X \times [0,1] \big/ \substack{(x,0) \sim (y,0) \\ \forall x,y \in X.}$$



Eg: Double dual

$$D^2 : \underline{Vec_k} \longrightarrow \underline{Vec_k}$$
$$V \longmapsto V^{**}$$

Not necessarily finite dimensional.

Exercise: Check this is a functor after specifying action on morphisms.

Natural Transformations

Suppose $\mathcal{C}, \mathcal{D}$ are categories,

$F, G : \mathcal{C} \to \mathcal{D}$ functors.

Def: A natural transformation $\delta : F \to G$

is a morphism $\delta_X : F(X) \to G(X)$ $\forall X \in Ob(\mathcal{C})$

st $\forall X, Y \in Ob(\mathcal{C})$, $f \in Mor(X, Y)$, the diagram commutes.

$$\begin{tikzcd} F(X) \arrow[r, "\delta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\delta_Y"] & G(Y) \end{tikzcd}$$

Exercise: The connecting homomorphism δ, with $\delta_{(X,A)} : H_k(X,A) \to H_{k-1}(A)$ is a natural transformation.

Eg: Define $Id_F : F \to F$ by

$Id_{F(X)} = Id_{F(X)} : F(X) \to F(X)$.

This is a natural transformation.

Def: We call $\delta : F \to G$ a natural isomorphism iff δ_X is an isom for all $X \in Ob(\mathcal{C})$.

Exercise: Let $D^2 : \underline{Vec_k} \to \underline{Vec_k}$ be the double dual.

Field, no longer a natural number

Then there is a natural transformation from $Id_{\underline{Vec}}$ to D^2

Note: Funny business when $V \in \underline{Vec_k}$ is infinite dimensional as $V \not\cong V^{**}$

Complexes

Def: A chain complex (of abelian groups) is a sequence

$C_* = (C_k, \partial_k)_{k \in \mathbb{Z}}$ with $C_k \in Ob(\underline{Ab})$

and $\partial_k : C_k \to C_{k-1}$ st $\partial_{k-1} \circ \partial_k = 0$ $\forall k \in \mathbb{Z}$.

Eg: Suppose $X \in \underline{Top}$, C_k^{sing} the free abelian group generated by singular simplices $\sigma : \Delta^k \to X$.

Define $\partial_k^{sing} \sigma = \sum_{i=0}^{k} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_k]}$ and extend linearly.

This gives $C_*^{sing} = (C_k^{sing}, \partial_k^{sing})$

Def: A morphism $f_* : C_* \to D_*$ of chain complexes is a sequence $f_* = (f_k : C_k \to D_k)$ st $\partial_k^D \circ f_k = f_{k-1} \circ \partial_k^C$ $\forall k \in \mathbb{Z}$.

i.e. The diagram
$$\begin{tikzcd} C_k \arrow[r, "\partial_k^C"] \arrow[d, "f_k"] & C_{k-1} \arrow[d, "f_{k-1}"] \\ D_k \arrow[r, "\partial_k^D"] & D_{k-1} \end{tikzcd}$$
commutes,

Eg: If $f : X \to Y$ is a continuous map of topological spaces, then $f_* : C_*^{sing}(X) \to C_*^{sing}(Y)$ is a chain map.

New categories from old.

Def: $\underline{Kom_{\mathbb{Z}}}$ is the category of chain complexes with morphisms given by chain maps.

Def: $\underline{Grad_{\mathbb{Z}}}$ is the category of graded abelian groups

What is homology?

$$\underline{Top} \xrightarrow{C^{sing}} \underline{Kom_{\mathbb{Z}}} \xrightarrow{H_k} \underline{Grad_{\mathbb{Z}}}$$
$$X \longmapsto C_*^{sing}(X) \longmapsto H_*^{sing}(X)$$