LECTURE 3

10. R - Modules

Suppose R is a commutative ring with identity 1_R.

Def: An R-module M is a quadruple $(M, +, \cdot, 0_M)$ such that $(M, +, 0)$ is an abelian group and $\cdot : R \times M \to M$ is an R-action (with usual axioms) ~~e.g~~
e.g. for $r, s \in R$ $m, n \in M$ we have
$1_R \cdot m = m$
$(r+s)m = rm + sm$
$r(m+n) = rm + rn$.

Examples: If R is a field then R-modules are vector spaces.
If $R = \mathbb{Z}$ then R-modules are abelian groups.
Hence the notation $\underline{Ab} = \underline{Mod}_{\mathbb{Z}}$.

Def: An R-module is free if it has a ~~[illegible]~~ basis B.
So: $M \cong \bigoplus_{b \in B} R$. But this isomorphism depends on the basis chosen so is not "natural". This splitting may be "unnatural".

Example: $\mathbb{Z}/2\mathbb{Z}$ is free as a $\mathbb{Z}/2\mathbb{Z}$-module but not as a $\mathbb{Z}$-module.

Exercise: $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.
Exercise: ~~Suppose~~ Find ~~[illegible]~~ R and M a free R-module such that there is $N < M$ (N a submodule of M) ~~[illegible]~~ which is not free.

Prop: Suppose R is a PID (principal ideal domain). Then submodules of finitely generated free R-modules are again free.

In fact we have

Thm (Classification of modules over PIDs): If R is a PID and M is finitely generated ~~free~~ then M is a direct sum of cyclic modules

$$M \cong \bigoplus_k R/I_k.$$

11. R-homomorphisms

Def: Suppose M, N are R-modules. Say $f: M \to N$ is ~~R-linear~~ R-linear if $f(rm + sn) = rf(m) + sf(n)$ for all $r, s \in R$ $m, n \in M$.

Define $\underline{\mathrm{Mod}}_R$ to be the category of R-modules and R-linear maps.

Exercise: Show that $\underline{\mathrm{Mod}}_R$ is indeed a category.

Notation: $\mathrm{Mor}(M, N) = \mathrm{Hom}_R(M, N)$ = R-linear maps $f: M \to N$.

Note: $\mathrm{Hom}_R(M, N)$ is given the structure of an R-module as follows:

$$(f+g)(m) = f(m) + g(m)$$
$$(r \cdot f)(m) = f(rm) \quad (R \text{ commutative}).$$
$$0_{\mathrm{Hom}} = \text{constant function } f(m) = 0_N.$$

Exercise: Show that $\mathrm{Hom}_R(M, N)$ is indeed an R-module.

Exercise: Fix $P \in \underline{\mathrm{Mod}}_R$. Define a map

$$N \xmapsto{F} \mathrm{Hom}_R(P, N).$$

This extends to morphisms: Given $f: M \to N$ then

$$F(f) : \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, N)$$
$$g \longmapsto f \circ g.$$

Then $F : \underline{\mathrm{Mod}}_R \to \underline{\mathrm{Mod}}_R$ is a functor.

Exercise: If $P = R$ (the cyclic R-module ~~R~~ ~~not~~ R not the ring R) then F is naturally isomorphic to $\mathrm{Id}_{\underline{\mathrm{Mod}}_R}$.

That is, $\mathrm{Hom}_R(R, N) \cong N$ and the isomorphism is natural.

Exercise: Note that fixing Q we also have ~~$M \mapsto \mathrm{Hom}_R$~~ $M \mapsto \mathrm{Hom}_R(M, Q)$. This is more interesting!

12. Contravariant

Def: Say $F : \mathcal{C} \to \mathcal{D}$ is a contravariant functor if $F(f \circ g) = F(g) \circ F(f)$ and $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$.

Exercise ~~Example~~: $M \mapsto \mathrm{Hom}_R(M, Q)$ is contravariant.

Here's the main example:

Notation: Fix Q. Define F_Q as before and write
$F_Q : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}^R$, $M \longmapsto M^* = \mathrm{Hom}_R(M, Q)$.
We call M^* the dual of M with coefficients in Q.
(Most of the time ~~as~~ we'll have $Q = \mathbb{Z}$).

Define $\underline{Kom^R}$ as follows:

$C^* = (C^k, \delta^k)_{k \in \mathbb{Z}}$, where C^k is an R-module and $\delta^k : C^{k-1} \longrightarrow C^k$ such that $\cancel{\delta^k \delta^{k+1}}$ $\delta^{k+1} \circ \delta^k = 0$, is a cochain complex, and $\underline{Kom^R}$ is the category of such.

Define $\underline{Grad^R}$ to be the category of graded R-modules. So $\underline{Grad^R}$ is equivalent to $\underline{Grad_R}$ as a category.

13. Cohomology

First a review

Singular homology:

$$\underline{Top} \longrightarrow \underline{Kom_R} \longrightarrow \underline{Grad_R}$$

$$X \longmapsto C_*^{sing} \longmapsto H_*^{sing}$$

Here $H_k = \dfrac{Z_k}{B_k} = \dfrac{\text{cycles}}{\text{boundaries}}$ for $Z_k = \ker(\partial_k)$
$B_k = \mathrm{im}(\cancel{\partial_k})$.
∂_{k+1}

$$\underline{Top} \longrightarrow \underline{Kom_R} \longrightarrow \underline{Kom^R} \longrightarrow \underline{Grad^R}$$

$$X \longmapsto C_*^{sing} \xrightarrow{F_Q} C^*_{sing} \longrightarrow H^*_{sing} = \frac{Z^*}{B^*}$$

for $Z^k = \ker(\delta^{k+1})$, $B^k = \mathrm{im}(\delta^k)$.

Notation: $\cdots \longrightarrow C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots$ a chain complex

dualising

$$\cdots \longleftarrow C^{k+1} \xleftarrow{\delta_{k+1}} C^k \xleftarrow{\delta_k} C^{k-1} \xleftarrow{\delta_{k-1}} \cdots$$ ~~cochain complex~~

Now take (co)homology to get $H^*(C)$.

<u>Moral</u>: The operations "take homology" and "dualise" do not commute:

$$
\begin{tikzcd}
 & C_* \arrow[dl, "\text{dualise}"'] \arrow[dr, "\text{take homology}"] & \\
{[\mathrm{Hom}_R(C_*, Q) =]\, C^*} \arrow[d, "\text{take hom}"'] & & H_* \arrow[d, "\text{dualise}"] \\
H^*(C) & \not\cong & (H_*)^* = \mathrm{Hom}(\cancel{H_R}\ H_*, Q)
\end{tikzcd}
$$

not isomorphic in general.