MA4J7 2022-01-17

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(14) Cohomology - Computed

Given

$$\begin{tikzcd}
\cdots \arrow[r] & C_{k+1} \arrow[r, "\partial_{k+1}"] \arrow[d, phantom, "\vee"] \arrow[ddr] & C_k \arrow[r, "\partial_k"] \arrow[d, phantom, "\vee"] \arrow[ddr] & C_{k-1} \arrow[r] \arrow[d, phantom, "\vee"] & \cdots \\
 & Z_{k+1} \arrow[d, phantom, "\vee"] \arrow[ddr] & Z_k \arrow[d, phantom, "\vee"] \arrow[ddr] & Z_{k-1} \arrow[d, phantom, "\vee"] & \\
{} \arrow[r] & B_{k+1} \arrow[d, phantom, "\vee"] & B_k \arrow[d, phantom, "\vee"] & B_{k-1} \arrow[d, phantom, "\vee"] & \\
{} \arrow[r] & 0 & 0 & 0 &
\end{tikzcd}$$

and $H_k = \dfrac{Z_k}{B_k}$ and we now dualise.

Suppose $Q \in \underline{\mathrm{Mod}_R}$, define $M^* = \mathrm{Hom}_R(M, Q)$.

Ex:

$$\begin{tikzcd}
\overset{2}{0} \arrow[r] & \overset{1}{\mathbb{Z}} \arrow[r, "\times 2"] \arrow[d, phantom, "\vee"] \arrow[ddr] & \overset{0}{\mathbb{Z}} \arrow[r] \arrow[d, phantom, "\vee"] \arrow[ddr] & \overset{-1}{0} \arrow[d, phantom, "\vee"] \\
 & 0 \arrow[d, phantom, "\vee"] \arrow[ddr] & \mathbb{Z} \arrow[d, phantom, "\vee"] \arrow[ddr] & 0 \arrow[d, phantom, "\vee"] \\
 & 0 \arrow[d, phantom, "\vee"] & 2\mathbb{Z} \arrow[d, phantom, "\vee"] & 0 \arrow[d, phantom, "\vee"] \\
 & 0 & 0 & 0
\end{tikzcd}$$

So $H_0 \cong \mathbb{Z}/2\mathbb{Z}$, $H_k \cong 0$ for $k \neq 0$.

Dualise ($Q = \mathbb{Z}$)

$$0^\vee \leftarrow \mathbb{Z}^\vee \xleftarrow{(\times 2)^\vee} \mathbb{Z}^\vee \leftarrow 0^\vee$$

$$0 \leftarrow \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \leftarrow 0$$

check : $(\times 2)^\vee (d)(k) = d(2k) = 2d(k)$

$\Rightarrow \quad (\times 2)^\vee (d) = 2d$.

So/

$$\begin{tikzcd} 0 & \mathbb{Z} \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l, "\times 2"'] \arrow[d] & 0 \arrow[l] \\ & \mathbb{Z} \arrow[d] & 0 \arrow[d] & \\ & 2\mathbb{Z} \arrow[d] & 0 \arrow[d] & \\ & 0 & 0 & \end{tikzcd}$$

<u>Thus:</u> $H^1 \cong \mathbb{Z}/2\mathbb{Z}$, $H^k \cong 0$ if $k \neq 1$.

So we see that

$$\begin{tikzcd} & C_\bullet \arrow[dl] \arrow[dr] & \\ C^\bullet \arrow[d] & & H_\bullet \arrow[d] \\ H^\bullet \arrow[rr, phantom, "\not\cong"] & & (H_\bullet)^\vee \end{tikzcd}$$

So we see that even if Q is free, $H^\bullet$ need not be iso to $(H_\bullet)^\vee$

Q: What if Q is an injective R-mod?
(i.e. Q is divisible?)

15 Universal coefficient theorem

<u>Thm</u>: Suppose R is a PID. Fix Q an R-mod. Suppose $C_n \in \underline{Kom_R}$ has C_k free for all $k \in \mathbb{Z}$. Then

$$0 \longrightarrow \mathrm{Ext}_R(H_{k-1}(C), Q) \to H^k(C,Q) \to \mathrm{Hom}_R(H_k(C), Q) \to 0$$

is a split s.e.s. Also, the UCT gives a contravariant functor from $\underline{Kom_R}$ to $\underline{SESGrad}^{R}$.

So if $f_\cdot : C_n \to D_n$ is a chain map then we have

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ext}_R(H_{k-1}(C), Q) & \to & H^k(C,Q) & \to & \mathrm{Hom}_R(H_k(C), Q) & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \mathrm{Ext}_R(H_{k-1}(D, Q)) & \to & H^k(D,Q) & \to & \mathrm{Hom}_R(H_k(D, Q)) & \to & 0 \end{array}$$

16 Sequences: Call $C_n \in \underline{Kom}_R$ exact (acyclic) if $H_n(C) \cong 0$.

<u>Equiv</u>: If $Z_k = B_k$ for all k.

A sequence C_n is (very(very)) short it has at most (1,2) 3 non-zero (and adjacent) terms.

Ex: $0 \to \mathbb{Z} \to 0$

$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to 0$

$0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$

$a \mapsto (a,0)$

$(a,b) \mapsto b$

Half-Time question:

What if we use the quaternions instead of R-modules?

Answer: No Idea (problem with left modules vs right modules)

Def: We say a s.e.s

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

is split (or splits) if any one of these properties hold:

1) There is a section $s: C \to B$ s.t. $j \circ s = id_C$

2) There is a projection $p: B \to A$ s.t. $p \circ i = id_A$

3) There is a factorisation $f: B \cong A \oplus C$

$$\begin{tikzcd}
& & B \arrow[dr, "j"] \arrow[d, "F"', "\cong"] & & \\
0 \arrow[r] & A \arrow[ur, "i"] \arrow[r] & A\oplus C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

$a \mapsto (a,0)$ $\qquad (a,c) \mapsto c$

Exercise/lemma (splitting lemma):
The conditions are all equivalent

Rmk: We say that

$$0 \to A \to B \to C \to 0$$

splits but it may do so in many ways!!!
(splitting may be unnatural)

Exercise: ~~If~~ $0 \to A \to B \to C \to 0$ is
short exact, and C is free then the sequence
splits (Ax of choice).

Question: Something about projective modules.
The second part of the splitting lemma

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lemma: Spose $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$
is s.e split. Then so is

$$0 \leftarrow A^\vee \xleftarrow{i^\vee} B^\vee \xleftarrow{j^\vee} C^\vee \leftarrow 0.$$

[so fix R, and $Q \in \underline{Mod}_R$

Prop: We must show

i) $i^\vee$ is surj
ii) $j^\vee$ is inj
iii) $\ker(i^\vee) = \operatorname{im}(j^\vee)$
iv) the dual seq. splits

i) We are given $p : B \to A$ s.t $p \circ i = Id_A$.
Dualise to find $i^\vee \circ p^\vee = Id_{A^\vee}$.
So $i^\vee$ is surjective (has a right inverse).

ii) Given $s : C \to B$ s.t $j \circ s = Id_C$.
So $s^\vee \circ j^\vee = Id_{C^\vee}$.
Hence $j^\vee$ is injective.

iii) Use previous lemma.
Since s is a section, $s^\vee$ is a projection.

iii) Next time.