

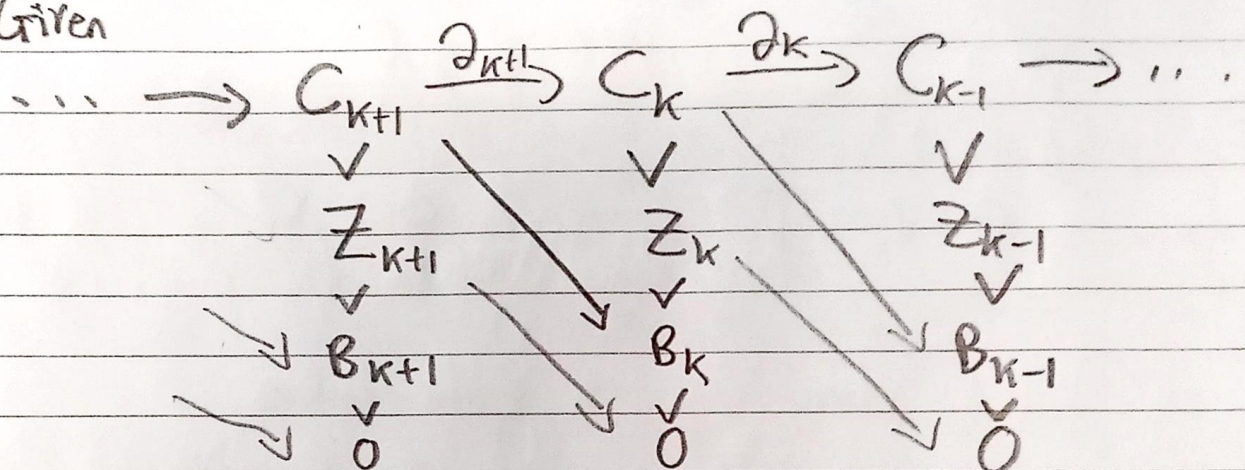
MAT J7 2022-01-17

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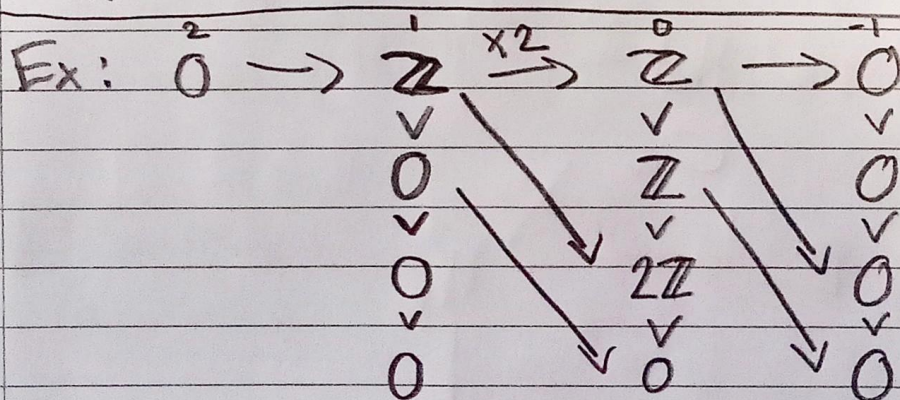
⑭ Cohomology - Computed

Given



and  $H_k = \frac{Z_k}{B_k}$  and we now dualize.

Suppose  $Q \in \text{Mod}_R$ , define  $M^k = \text{Hom}_R(M, Q)$ .



So  $H_0 \cong \mathbb{Z}/2\mathbb{Z}$ ,  $H_k \cong 0$  for  $k \neq 0$ .



Dualise ( $Q = \mathbb{Z}$ )

$$0^n \leftarrow \mathbb{Z}^n \xleftarrow{(x_2)^n} \mathbb{Z}^n \leftarrow 0^n$$

$$0 \leftarrow \mathbb{Z} \xleftarrow{x_2} \mathbb{Z} \leftarrow 0$$

check:  $(x_2)^n(d)(k) = d(2k) = 2d(k)$

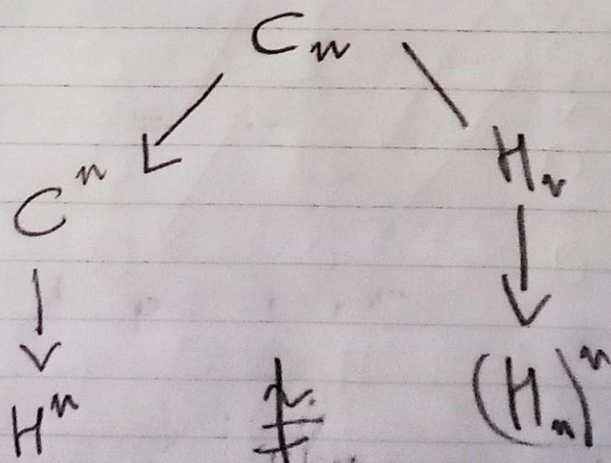
$$\Rightarrow (x_2)^n(d) = 2d$$

So

$$\begin{array}{ccccccc}
 0 & \leftarrow & \mathbb{Z} & \xleftarrow{x_2} & \mathbb{Z} & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbb{Z} & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & 2\mathbb{Z} & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

Thus:  $H^1 \cong \mathbb{Z}/2\mathbb{Z}$ ,  $H^k \cong 0$  if  $k \neq 1$ .

So we see that



So we see that even if  $Q$  is free,  $H^n$  need not be iso to  $(H_n)^n$



Q: What if  $Q$  is an injective  $R$ -mod?  
 (if  $Q$  is divisible?)

15) Universal coefficient theorem

Thm: Suppose  $R$  is a PID. Fix  $C$  an  $R$ -mod. Suppose  $C_n \in \text{Kom}_R$  has  $C_k$  free for all  $k \in \mathbb{Z}$ . Then

$$0 \longrightarrow \text{Ext}_R(H_{k+1}(C), Q) \longrightarrow H^k(C, Q) \longrightarrow \text{Hom}_R(H_k(C), Q) \longrightarrow 0$$

is a split s.e.s. Also, the UCT gives a contravariant functor from  $\text{Kom}_R$  to  $\text{SES Grad}$ .

So if  $f_n: C_n \rightarrow D_n$  is a chain map then we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ext}_R(H_{k+1}(C), Q) & \longrightarrow & H^k(C, Q) & \longrightarrow & \text{Hom}_R(H_k(C), Q) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \downarrow \\ 0 & \longrightarrow & \text{Ext}_R(H_{k+1}(D), Q) & \longrightarrow & H^k(D, Q) & \longrightarrow & \text{Hom}_R(H_k(D), Q) \longrightarrow 0 \end{array}$$

16) Sequences: Call  $C_n \in \text{Kom}_R$  exact (acyclic) if  $H_n(C) \cong 0$ .

Equiv: If  $Z_k = B_k$  for all  $k$ .

A sequence  $C_n$  is (very very) short if it has at most 3 non-zero (and adjacent) terms.



Ex:  $0 \rightarrow \mathbb{Z} \rightarrow 0$   
 $0 \rightarrow \mathbb{Z} \xrightarrow{x^2} \mathbb{Z} \rightarrow 0$   
 $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z} \rightarrow 0$   
 $a \mapsto (a, 0)$   
 $(a, b) \mapsto b$

Half-Time question:

What if we use the quaternions instead of  $R$ -modules?

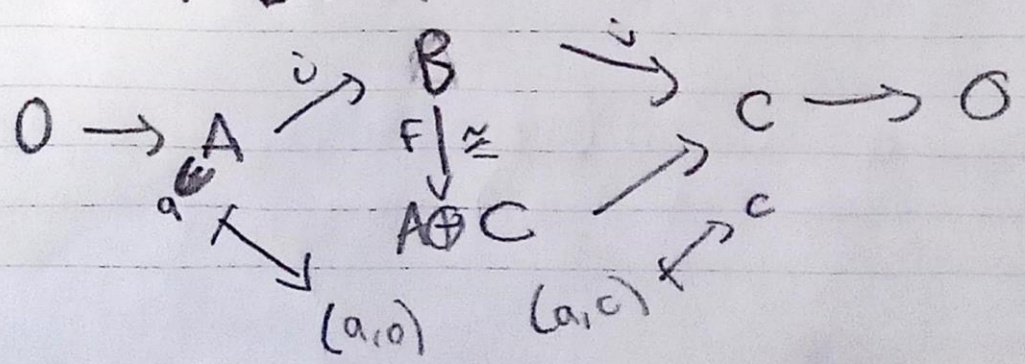
Answer: No Idea (problem with left modules vs right modules)

Def: We say a ~~short~~ s.e.s.  $\pi$

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$$

is split (or splits) if any one of those properties hold:

- 1) There is a section  $s: C \rightarrow B$  s.t.  $jos = id_C$
- 2) There is a projection  $p: B \rightarrow A$  s.t.  $poi = id_A$
- 3) There is a factorization  $f: B \cong A \oplus C$





Exercise/lemma (splitting lemma):  
The conditions are all equivalent

Rmk: We say that

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

splits but it may do so in many ways!!!

(splitting may be unnatural)

Exercise:  $\nLeftrightarrow 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is

short exact, and  $C$  is free then the sequence splits (A is of choice).

Question: Something about projective modules.

The second part of the splitting lemma

lemma: Suppose  $0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$   
is s.e. split. Then so is

$$0 \leftarrow A'' \xleftarrow{i''} B'' \xleftarrow{j''} C'' \leftarrow 0$$

[So fix  $R$ , and  $\mathcal{A} \in \text{Mod}_R$

Prop: We must show

i)  $i''$  is surj

ii)  $j''$  is inj

iii)  $\ker(i'') = \text{im}(j'')$

iv) the dual seq. splits



i) We are given  $p: B \rightarrow A$  s.t.  $p \circ i = \text{Id}_A$ .  
Dualise to find  $i^{\vee} \circ p^{\vee} = \text{Id}_{A^{\vee}}$ .  
So  $i^{\vee}$  is surjective (has a right inverse).

ii) Given  $s: C \rightarrow B$  s.t.  $j \circ s = \text{Id}_C$ .  
So  $s^{\vee} \circ j^{\vee} = \text{Id}_{C^{\vee}}$ .  
Hence  $j^{\vee}$  is injective.

iii) Use previous lemma.  
Since  $s$  is a section,  $s^{\vee}$  is a projection.

ii) Next time.