MA4J7 - Cohomology — 18th January 2022

Stephen Richardson

<u>Lemma</u>: Suppose $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ is short, split, exact. Then:
$0 \longleftarrow A^* \xleftarrow{i^*} B^* \xleftarrow{j^*} C^* \longleftarrow 0$ is also short, split, exact.

Proof (Continued): (Could possibly use universal property of cokernels)

iii) Exactness at B^*

Want to show that $\mathrm{Ker}(i^*) = \mathrm{Im}(j^*)$

Fix $\beta \in \mathrm{Im}(j^*)$

That is, there is some $\gamma \in C^*$ such that $j^*(\gamma) = \beta$.

$C^* = \mathrm{Hom}_R(C, Q)$

<u>Thus</u>: $j^*(\gamma) = \gamma \circ j \in \mathrm{Hom}_R(B, Q) = B^*$

<u>We compute</u>:
$$\begin{aligned} i^*(\beta) &= i^*(j^*(\gamma)) \\ &= (i^* \circ j^*)(\gamma) \\ &= (j \circ i)^*(\gamma) \\ &= 0^*(\gamma) = 0 \end{aligned}$$

So $\beta \in \mathrm{Ker}(i^*)$, $\therefore \mathrm{Im}(j^*) \leq \mathrm{Ker}(i^*)$

(Could just use $i^* \circ j^* = (j \circ i)^* = 0$ directly)

Now fix $\beta \in \mathrm{Ker}(i^*)$.

So $i^*(\beta) = 0$, $\therefore \beta \circ i = 0$

Thus $\beta|_{\mathrm{Im}(i)} = 0$

So by exactness, $\beta|_{\mathrm{Ker}(j)} = 0$

Define $\gamma \in C^* = \mathrm{Hom}_R(C, Q)$ as follows:

$\gamma(c) = \beta(b)$ for $b \in j^{-1}(c)$

<u>Check</u>: 1) γ is well-defined

Proof: Suppose $b', b \in j^{-1}(c)$

Then $j(b - b') = j(b) - j(b') = c - c = 0$

$\therefore b - b' \in \mathrm{Ker}(j)$

So by above, $\beta(b) - \beta(b') = \beta(b - b') = 0$

$\therefore \beta(b) = \beta(b')$, and γ is well defined

2) γ is R-linear

Proof: β is R-linear. Pick $b \in j^{-1}(c)$ and $b' \in j^{-1}(c')$

So: $j(rb + sb') = rc + sc'$

Thus $rb + sb' \in j^{-1}(rc + sc')$

So $\gamma(rc + sc') = \beta(rb + sb')$

$= r\beta(b) + s\beta(b')$

$= r\gamma(c) + s\gamma(c')$ $\square$

Exercise: Check $\gamma(c) = \beta(s(c))$ for a section $s : C \to B$

Exercise: The functor $Hom_R(\circ, Q)$ is right-exact.
That is, if $A \to B \to C \to 0$ is exact,
then so is $A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$

Example: $R = \mathbb{Z} = Q$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0 \qquad (m \neq 0)$$

Dualise to get:

exact

$$0 \leftarrow \mathbb{Z} \xleftarrow{\times m} \mathbb{Z} \leftarrow 0 \leftarrow 0$$

Not exact

17 - Ext

Morally Ext measures the failure of left exactness of $Hom_R(\circ, Q)$

(By Ext we mean $Ext_R(\circ; Q)$)

Free Resolutions

Suppose $M \in \underline{Mod_R}$. A sequence $F_* \in \underline{Kom_R}$ is a free resolution of M if:

i) $F_{-1} = M$

ii) $F_k = 0$ for $k < -1$

iii) F_k is free for $k > -1$

iv) F_* is exact

$$\cdots \to F_3 \to F_2 \to F_1 \to F_0 \to M$$

F_0 = Module of generators
F_1 = Module of relations
F_2 = Module of Syzygies
F_K for $K \geq 3$ are the higher Syzygies

<u>Example</u>:
$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$
is a free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $R = \mathbb{Z}$.

<u>Exercise</u>: If $R = \mathbb{F}$ is a field then there exists a free resolution $F_\bullet$ of M with $F_K = 0$ $\forall K \geq 1$

<u>Exercise</u>: If $R = \mathbb{Z}$ and M is finitely generated, then there exists an $F_\bullet$ with $F_K = 0$ $\forall K \geq 2$

<u>Exercise</u>: Find R, M such that any free resolution has infinite length.

<u>Exercise</u>: Find free resolution of $\mathbb{Q}$ over $R = \mathbb{Z}$
<u>Easier</u>: $M = \mathbb{Z}[\frac{1}{2}]$

<u>Lemma</u>: Fix R. Then free resolutions exist.

<u>Proof</u>: Fix M.
<u>Base Case</u>: Define $F_0 = \bigoplus_{m \in M} R e_m$

and
$$f_0 : F_0 \longrightarrow M$$
$$e_m \longmapsto m$$

In general, given $f_K : F_K \longrightarrow F_{K-1}$, define $F_{K+1} = \bigoplus_{m \in \ker(f_K)} R e_m$, and $f_{K+1}(e_m) = m$.

$\square$

Def^n: Suppose $M \in \underline{Mod}_R$. Let F_* be the free resolution as in the lemma.

Define $Ext_R^*(M;Q) := H^*(F;Q)$.

In particular, we define the "shorthand"

$Ext_R(M;Q) := Ext_R^1(M;Q) = H^1(F;Q)$

Remark / Diagram:

$$\cdots \to F_2 \to F_1 \to F_0 \xrightarrow{f_0} M \to 0$$

Dualise to get

$$\leftarrow F^2 \leftarrow F^1 \leftarrow F^0 \xleftarrow{f_0^*} M^* \leftarrow 0$$

We are exact at the right, so $H^0(F,Q) = Ext^0(M;Q)$ is trivial.

(f_0 is surjective, so f_0^* is injective - check)

Note: F^* is not exact in general.