

MA4J7 - Cohomology & Poincaré Duality

Lecture 6 - 20th January 2022

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Ext: Why the name ext? - stands for "extension" - broadly speaking, it is the set of equivalence classes of exact sequences:

$$0 \rightarrow Q \rightarrow A_{k-1} \rightarrow A_{k-2} \rightarrow \dots \rightarrow A_0 \rightarrow M \rightarrow 0$$

- Refer to "Homology" by Saunders-MacLane for a more entire runthrough.

Def: R -ring, $M, Q \in \text{Mod } R$. Let F_* be the tautological free resolution of M , so

$$\rightarrow F_k = 0 \text{ for } k < -1;$$

$$\rightarrow F_k = M;$$

$$\rightarrow f_{-1}: F_{-1} \rightarrow F_{-2} \text{ is zero.}$$

- Broadly speaking, F_{k+1} is the free R -module generated by $\text{Ker}(f_k)$, where

$$f_{k+1}: F_{k+1} \rightarrow F_k \text{ sends}$$

$$\sum_{i \in F_{k+1}} r_i m_i \text{ to } \sum_{i \in F_k} r_i m_i$$

Now define $\text{ext}_R^k(M, Q) := H^k(F; Q)$.

Exercise: Prove the following:

$$(1) \text{ext}^k(M, Q) \cong 0 \text{ for } k \leq 0.$$

$$(2) \text{ext}^k(A \oplus B, Q) \cong \text{ext}^k(A, Q) \oplus \text{ext}^k(B, Q).$$

$$(3) \text{ext}^1(\mathbb{Z}/n\mathbb{Z}, Q) \cong Q/nQ.$$

- note that it's impossible to prove (2) & (3) from the definition.

• Lemma 3.1: Fix R -ring, and $M, N, Q \in \text{Mod}_R$.

Suppose that $\alpha: M \rightarrow N$ is some homomorphism.

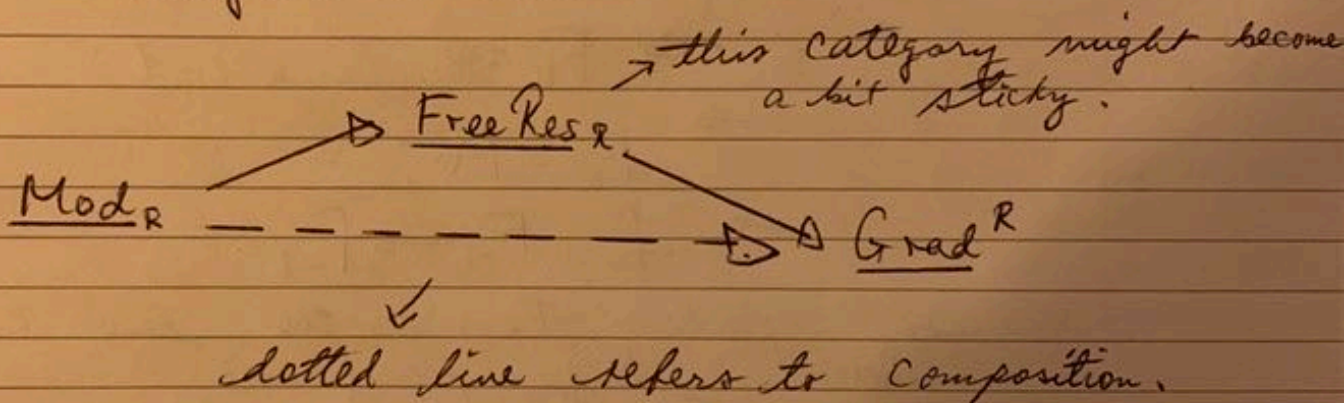
(A) Suppose F_* and G_* are any free resolutions of M and N respectively. Then \exists chain map $\alpha_*: F_* \rightarrow G_*$ such that $\rightarrow \alpha_{-1} = \alpha$;

\rightarrow If α_* & β_* are two such maps, then $\alpha_* \simeq \beta_*$ (Chain homotopic).

(B) If $M = N$ & $\alpha = \text{Id}_M$,

then $H^*(F, Q) \simeq H^*(G, Q)$ via ~~natural~~ natural isomorphism.

• Paul says: in practice, it yields a similar diagram to this:



There are functors as above, however there are issues making funct into a map on morphisms.

• Midpoint question: Doesn't this process merely shift the necessity of choosing/making a choice down a diagram a bit rather than solving the original problem? - loosely speaking, this method shifts the choice to a more appropriate point on a diagram (see below).

• Proof of 3.1: (A1) proceed recursively/inductively:

$$\begin{array}{ccccccc}
 \text{Base case: } & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & M & \xrightarrow{f_{-1}} & 0 \\
 & & & \downarrow & \circ & \downarrow \alpha & & \\
 & G_1 & \xrightarrow{g_1} & G_0 & \xrightarrow{g_0} & N & \xrightarrow{g_{-1}} & 0
 \end{array}$$

- Fix a basis of F_0 , and say $u \in F_0$ is a basis element.

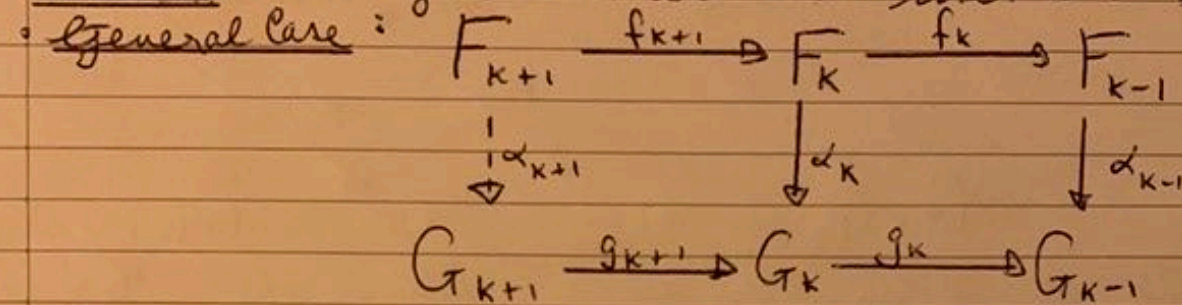
- note: $\alpha(f_0(u)) \in \text{Ker}(g_{-1})$, so by exactness, $\alpha(f_0(u)) \in \text{Im}(g_0)$.

- pick $v \in G_0$ s.t. $g_0(v) = \alpha(f_0(u))$, and set $\alpha_0(u) = v$ s.t. the square commutes.

- Thus $\alpha_{-1} \circ f_0 = g_0 \circ \alpha_0$.

beginning of a chain map.

• Remark: Choosing basis elements uses the axiom of choice



- pick a basis for F_{k+1} , with u a basis element.

note $\alpha_{k-1} f_k f_{k+1}(u) = 0$ since $f^2 = 0$, so by inductive hypothesis:

$$g_k \alpha_k f_{k+1}(u) = 0,$$

yielding commutativity.

- So $\alpha_k f_{k+1}(u) \in \text{Ker}(g_k) = \text{Im}(g_{k+1})$.

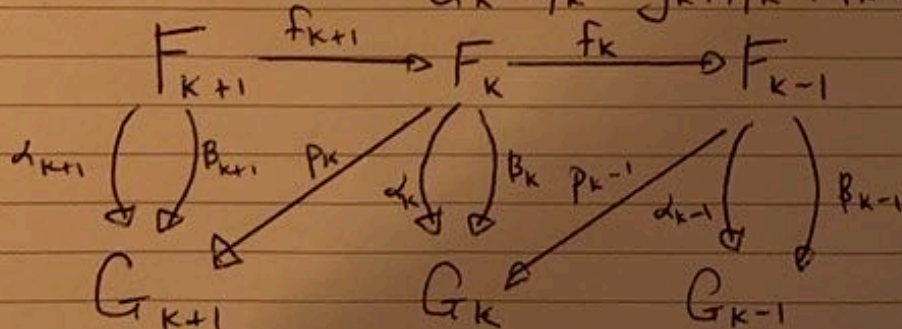
- pick $v \in G_{k+1}$ s.t. $g_{k+1}(v) = \alpha_k f_{k+1}(u)$.

- Define $\alpha_{k+1}(u) = v$. (A4)

• (A2) Recall that α_* and β_* are chain homotopic if there is a sequence of homomorphisms

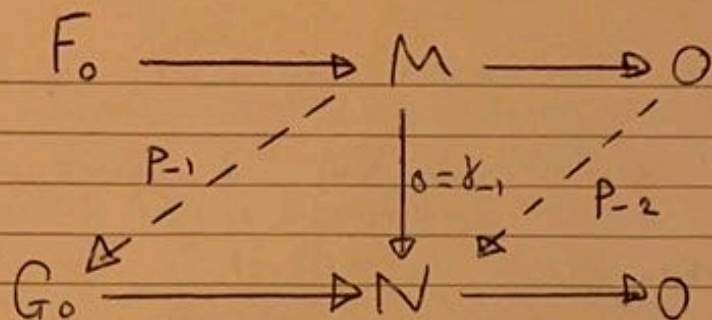
$$P_k: F_k \rightarrow G_{k+1} \text{ s.t.}$$

$$\alpha_k - \beta_k = g_{k+1} P_k + P_{k-1} f_k, \text{ via the diagram:}$$

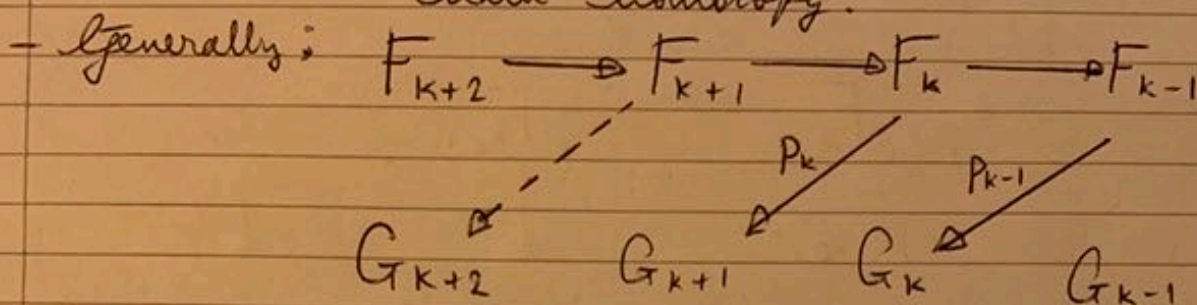


- P_k is a prism operator.

• Define $\gamma_* = \alpha_* - \beta_*$ (since homomorphisms are \mathbb{R} -linear).



- Should like to define $P_{-1} = 0$ as the base case.
- In general, need to check the zero map satisfies $(0\text{-map}) - (0\text{-map})$ st. it satisfies the chain homotopy.



- pick a basis for F_{k+1} , with a basis element $u \in F_{k+1}$.
- We would like: $\delta_{k+1}(u) = \delta_{k+2} P_{k+1}(u) + P_k f_{k+1}(u)$,

i.e. $\delta_{k+2} P_{k+1}(u) = \delta_{k+1}(u) - P_k f_{k+1}(u)$.

- Hence we want some v st.

$$\delta_{k+2}(v) = \delta_{k+1}(u) - P_k f_{k+1}(u)$$

i.e. is $\delta_{k+1}(u) - P_k f_{k+1}(u)$ in $\text{im}(\delta_{k+2})$?

- Question from student - why always pick basis elements?

- Answer - they always exist since we're dealing with free modules, and they can be extended linearly via the usual R -linear combinations.