MA4J7 – Cohomology & Poincaré Duality

Lecture 6 – 20th January 2022

Saul Schleimer

$\boxed{17}$ **Ext**: Why the name ext? – stands for "extension" – broadly speaking, it is the set of equivalence classes of exact sequences:

$$0 \to Q \to A_{k-1} \to A_{k-2} \to \cdots \to A_0 \to M \to 0.$$

- Refer to "Homology" by Saunders-MacLane for a more entire runthrough.

- **Def:** R – ring, $M, Q \in \underline{\mathrm{Mod}}_R$. Let F_* be the <u>tautological free resolution</u> of M, so
  - $F_k = 0$ for $k < -1$;
  - $F_k = M$;
  - $f_{-1} : F_{-1} \to F_{-2}$ is zero.

- Broadly speaking, F_{k+1} is the free R-module generated by $\mathrm{Ker}(f_k)$, where

$$f_{k+1} : F_{k+1} \to F_k \text{ sends } \underset{\text{in } F_{k+1}}{\sum r_i m_i} \text{ to } \underset{\text{in } F_k}{\sum r_i m_i}.$$

- Now define $\mathrm{ext}^k_R(M, Q) := H^k(F; Q)$.

- *Exercises*: Prove the following:

(1) $\mathrm{ext}^k(M, Q) \cong 0$ for $k \leq 0$.

(2) $\mathrm{ext}^k(A \oplus B, Q) \cong \mathrm{ext}^k(A, Q) \oplus \mathrm{ext}^k(B, Q)$.

(3) $\mathrm{ext}^1(\mathbb{Z}/n\mathbb{Z}, Q) \cong Q/nQ$.

- Note that it's impossible to prove (2) & (3) from the definition.

• <u>Lemma 3.1</u>: Fix R – ring, and $M, N, Q \in \underline{\mathrm{Mod}_R}$. Suppose that $\alpha : M \to N$ is some homomorphism.

(A) Suppose F_* and G_* are any free resolutions of M and N respectively. Then $\exists$ chain map $\alpha_* : F_* \to G_*$ such that

→ $\alpha_{-1} = \alpha$;

→ If α_* & β_* are two such maps, then $\alpha_* \simeq \beta_*$ (chain homotopic).

(B) If $M = N$ & $\alpha = \mathrm{Id}_M$, then $H^*(F, Q) \cong H^*(G, Q)$ via ~~natural~~ natural isomorphism.

• <u>Saul says</u>: in practice, it yields a similar diagram to this:

$$\begin{tikzcd}
 & \underline{\mathrm{FreeRes}_R} \arrow[dr] & \\
\underline{\mathrm{Mod}_R} \arrow[ur] \arrow[rr, dashed] & & \underline{\mathrm{Grad}^R}
\end{tikzcd}$$

this category might become a bit sticky.

dotted line refers to composition.

- There are functors as above, <u>however</u> there are issues making <u>taut</u> into a map on morphisms.

• <u>Midpoint question</u>: Doesn't this process merely shift the necessity of choosing/making a choice down a diagram a bit rather than solving the original problem?

– Loosely speaking, this method shifts the choice to a more appropriate point on a diagram (see below).

• <u>proof of 3.1</u>: (A1) proceed recursively/inductively:

<u>Base case</u>:

$$\begin{tikzcd}
F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] \arrow[d, dashed] \arrow[dr, phantom, "\circ"] & M \arrow[r, "f_{-1}"] \arrow[d, "\alpha"] & 0 \\
G_1 \arrow[r, "g_1"] & G_0 \arrow[r, "g_0"] & N \arrow[r, "g_{-1}"] & 0
\end{tikzcd}$$

- Fix a basis of F_0, and say $u \in F_0$ is a basis element.
- ~~note~~: $\alpha(f_0(u)) \in \ker(g_{-1})$, so by exactness,
$$\alpha(f_0(u)) \in \operatorname{im}(g_0).$$
- pick $v \in G_0$ s.t. $g_0(v) = \alpha(f_0(u))$, and set $\alpha_0(u) = v$ s.t. the square commutes.
- Thus $\underbrace{\alpha_{-1} \circ f_0 = g_0 \circ \alpha_0}_{\text{beginning of a chain map.}}$

• Remark: Choosing basis elements uses the axiom of choice

• General Case:
$$\begin{tikzcd}
F_{k+1} \arrow[r, "f_{k+1}"] \arrow[d, dashed, "\alpha_{k+1}"] & F_k \arrow[r, "f_k"] \arrow[d, "\alpha_k"] & F_{k-1} \arrow[d, "\alpha_{k-1}"] \\
G_{k+1} \arrow[r, "g_{k+1}"] & G_k \arrow[r, "g_k"] & G_{k-1}
\end{tikzcd}$$

- pick a basis for F_{k+1}, with u - a basis element. note $\alpha_{k-1} f_k f_{k+1}(u) = 0$ since $f^2 = 0$, so by inductive hypothesis:
$$g_k \alpha_k f_{k+1}(u) = 0,$$
yielding commutativity.
- So $\alpha_k f_{k+1}(u) \in \ker(g_k) = \operatorname{im}(g_{k+1})$
- pick $v \in G_{k+1}$ s.t. $g_{k+1}(v) = \alpha_k f_{k+1}(u)$.
- Define $\alpha_{k+1}(u) = v$. (A4)

• (A2) Recall that α_* and β_* are chain homotopic if there is a sequence of homomorphisms
$$P_k : F_k \longrightarrow G_{k+1} \quad \text{s.t.}$$
$\alpha_k - \beta_k = g_{k+1} P_k + P_{k-1} f_k$, via the diagram:
$$\begin{tikzcd}
F_{k+1} \arrow[r, "f_{k+1}"] \arrow[d, bend right, "\alpha_{k+1}"'] \arrow[d, bend left, "\beta_{k+1}"] & F_k \arrow[r, "f_k"] \arrow[dl, "P_k"'] \arrow[d, bend right, "\alpha_k"'] \arrow[d, bend left, "\beta_k"] & F_{k-1} \arrow[dl, "P_{k-1}"'] \arrow[d, bend right, "\alpha_{k-1}"'] \arrow[d, bend left, "\beta_{k-1}"] \\
G_{k+1} & G_k & G_{k-1}
\end{tikzcd}$$

- P_k is a <u>prism operator</u>.

• Define $\gamma_* = \alpha_* - \beta_*$ (since homomorphisms are R-linear).

$$\begin{tikzcd}
F_0 \arrow[r] & M \arrow[r] \arrow[d, "0=\gamma_{-1}"] \arrow[dl, dashed, "p_{-1}"'] & 0 \arrow[dl, dashed, "p_{-2}"] \\
G_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

- Would like to define $p_{-1}=0$ as the base case.
- In general, need to check the zero map satisfies (0-map) − (0-map) s.t. it satisfies the chain homotopy.
- Generally:

$$\begin{tikzcd}
F_{k+2} \arrow[r] & F_{k+1} \arrow[r] \arrow[dl, dashed] & F_k \arrow[r] \arrow[dl, "p_k"'] & F_{k-1} \arrow[dl, "p_{k-1}"'] \\
G_{k+2} & G_{k+1} & G_k & G_{k-1}
\end{tikzcd}$$

- Pick a basis for F_{k+1}, with a basis element $u \in F_{k+1}$.
- We would like: $\gamma_{k+1}(u) = g_{k+2}p_{k+1}(u) + p_k f_{k+1}(u)$,

  i.e. $$g_{k+2}p_{k+1}(u) = \gamma_{k+1}(u) - p_k f_{k+1}(u).$$
- Hence we want some v s.t.
  $$g_{k+2}(v) = \gamma_{k+1}(u) - p_k f_{k+1}(u),$$
  i.e. is $\gamma_{k+1}(u) - p_1 f_{k+1}(u)$ in $\operatorname{im}(g_{k+2})$?

• Question from student – why always pick basis elements?
- Answer – they always exist since we're dealing with free modules, and they can be extended linearly via the usual R-linear combinations.